CSE 312 Foundations II

2. Counting

Winter 2017
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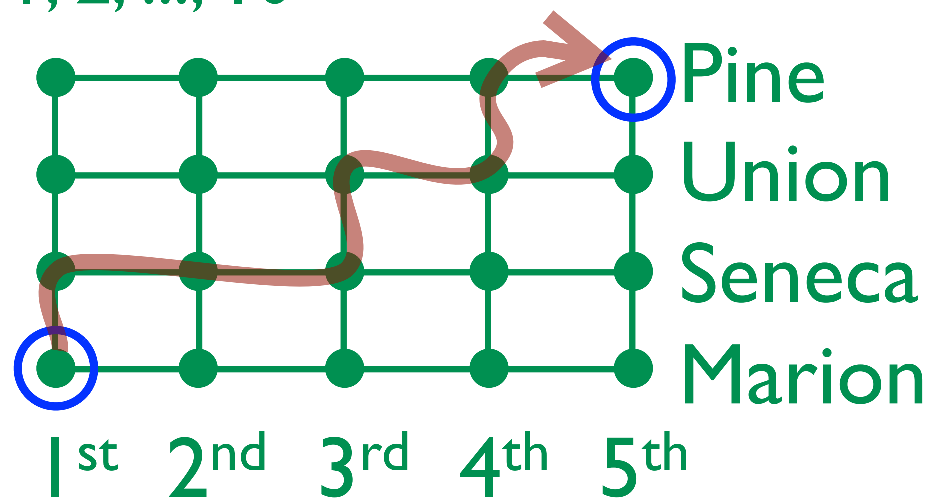
How many ways are there to do X?

E.g., X = “choose an integer 1, 2, ..., 10”

E.g., X = “Walk from 1st & Marion to 5th & Pine, going only North or East at each intersection.”



<u>The Point:</u>

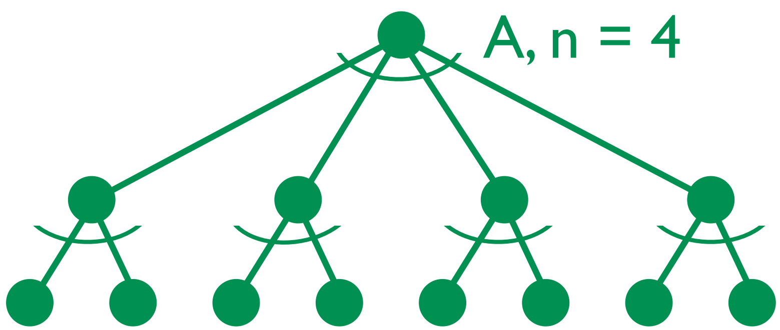
Counting gets hard when numbers are large, implicit and/or constraints are complex.

Systematic approaches help.

If there are

n outcomes for some event A,
sequentially followed by *m* outcomes for event B,

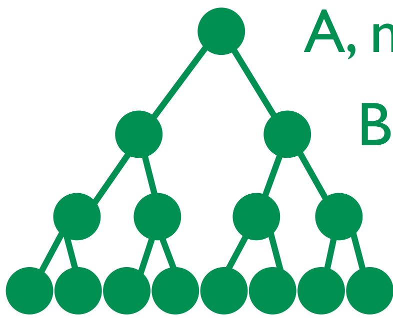
then there are *n•m* outcomes overall.



aka "The Product Rule"
Easily generalized to more events

Q. How many n-bit numbers are there?

A. 1st bit 0 or 1, then 2nd bit 0 or 1, then ...



A, $n_1 = 2$

B, $n_2 = 2$

C, $n_3 = 2$

$$\overbrace{2 \cdot 2 \cdot \ldots \cdot 2}^{n} = 2^n$$

Q. How many subsets of a set of size n are there?

A. 1st member in or out; 2nd member in or out,... ⇒ 2^n

> Tip: Visualize an order in which decisions are being made

Q. How many 4-character passwords are there, if each character must be one of a, b, ..., z, 0, 1, ..., 9 ?

A. $36 \cdot 36 \cdot 36 \cdot 36 = 1{,}679{,}616 \approx 1.7$ million

Q. Ditto, but no character may be repeated?

A. $36 \cdot 35 \cdot 34 \cdot 33 = 1{,}413{,}720 \approx 1.4$ million

(And a non-mathematical question: why do security experts generally prefer schemes such as the second, even though it offers fewer choices?)

Q. How many arrangements of 1, 2, 3 are possible (each used once, no repeat, order matters)?

1 2 3	2 1 3	3 1 2
1 3 2	2 3 1	3 2 1

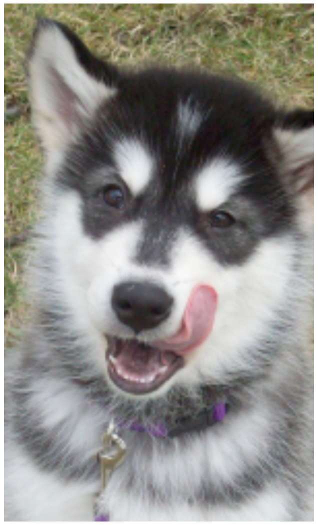
A. 3 • 2 • 1 = 6

Q. More generally: How many arrangements of n distinct items are possible?

n	choices for 1st
(n-1)	choices for 2nd
(n-2)	choices for 3rd
...	...
1	choices for last

A. n • (n-1) • (n-2) • ... • 1 = n! (n factorial permutations of n things)

Fine print: 0! = 1



Q. How many permutations of DAWGY are there?

A. $5! = 120$

Q. How many of DAGGY

A. $5!/2! = 60$

$DAG_1G_2Y = DAG_2G_1Y$
$ADG_1YG_2 = ADG_2YG_1$
...

Q. How many of GODOGGY ?

A. $$\frac{7!}{3!2!1!1!} = 420$$

3G's 2O's 1D 1Y

Q. Your elf-lord avatar can carry 3 objects chosen from

1. sword
2. knife
3. staff
4. water jug
5. iPad w/magic WiFi

How many ways can you equip him/her?

ordered ways in which to pick objects

A. $$\frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3! \cdot 2!} = 10$$

but picking abc is equiv to acb, and bca, and ...

"5 choose 3" ways to choose 3 objects from 5 possibilities

Combinations: number ways to choose r things from n

"n choose r" aka binomial coefficients

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1} = \frac{n!}{r!(n-r)!}$$

Middle formula:

n possibilities for 1st, (n-1) for 2nd, … (n-(r-1)) for rth, but that counts r! different choice orders for the same r objects.

Right formula:

Some algebra: Middle-top is the start of n!, but it's missing exactly the terms of (n-r)!

Combinations: number ways to choose r things from n

"n choose r" aka binomial coefficients

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1} = \frac{n!}{r!(n-r)!}$$

Right formula, viewed another way:

Write down all n objects in some order, then draw a line after the r^{th}; "choose" those to the left of the line. There are n! ways to write down the list, but each resulting set of r appears r!(n-r)! times in that list because there are r! ways to reorder the chosen objects left of the line and, independently, (n-r)! ways to reorder the unchosen objects to the right of the line.

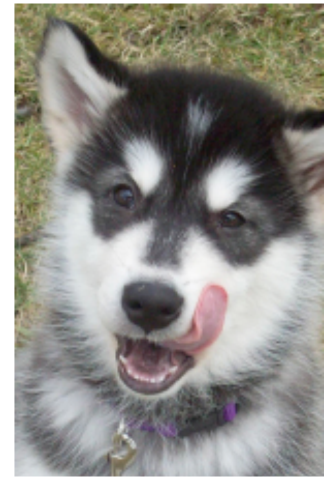
Try to find 2 ways to do every problem!

Convince yourself that you get the same answer

Which is easier to think of? To calculate? More general? Easier to explain? Why?

(You won't always succeed, but it's good exercise!)

Q. How many permutations of GODOGGY are there?



A. $\dfrac{7!}{3!2!1!1!} = 420$

View #1: Imagine subscripts on the letters so they are different; 7! orders. But for each placement of the G's and O's, there are 3!•2! different orderings of the subscripts, all giving identical words after the subscripts are removed:

$$G_3O_1O_2DYG_1G_2 = G_3O_2O_1DYG_1G_2 = G_3O_1O_2DYG_2G_1 = \ldots$$

View #2: 7 slots: _ _ _ _ _ _ _ ; 7 choose 3 slots to put G's; 4 choose 2 (remaining) slots to put O's; 2 choose 1 slots for D; 1 choose 1 slots for Y:

$$\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!}$$

Does it matter that I chose G's first, etc.?

Combinations: number ways to choose r things from n

"n choose r" aka binomial coefficients

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1} = \frac{n!}{r!(n-r)!}$$

Important special case:

how many (unordered) pairs from n objects

$$\binom{n}{2} = \frac{n(n-1)}{2} = \Theta(n^2)$$

Many Identities. E.g.:

$$\binom{n}{r} = \binom{n}{n-r}$$ ← by symmetry of definition

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$ ← first object either in or out; disjoint cases add

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$ ← by definition + algebra

Q. How many different poker hands are possible (i.e., 5 cards chosen from a deck of 52 distinct possibilities)?

A. $$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960$$

Q. 10 people meet at a party. If everyone shakes hands with everyone else, how many handshakes happen?

A. $$\binom{10}{2} = \frac{10 \cdot 9}{2 \cdot 1} = 45$$

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$

proof 1: induction ...

proof 2: counting –

(x+y) • (x+y) • (x+y) • ... • (x+y)

Eliminate parens via distributive law, etc.

pick either x or y from 1st binomial factor

pick either x or y from 2nd binomial factor

...

pick either x or y from nth binomial factor

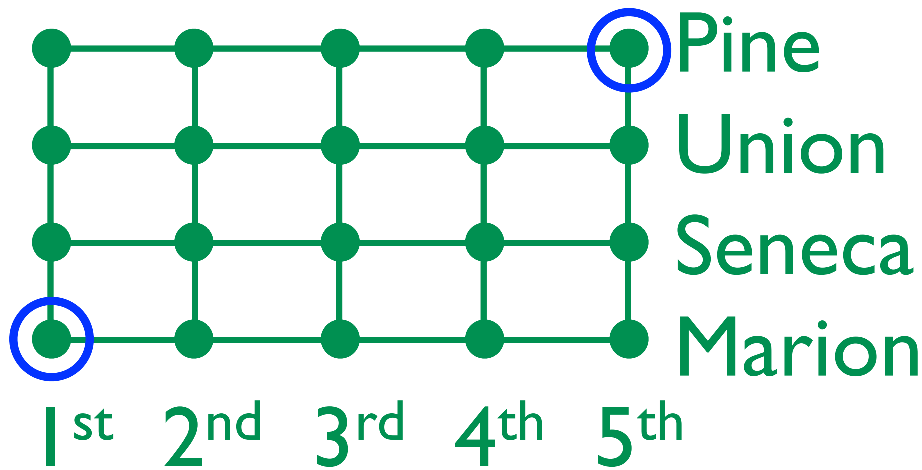
How many ways did you get exactly k x's? $\binom{n}{k}$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

Q. How many ways are there to walk from 1[st] & Marion to 5[th] & Pine, going only North or East?



A: 7 choose 3 = 35:

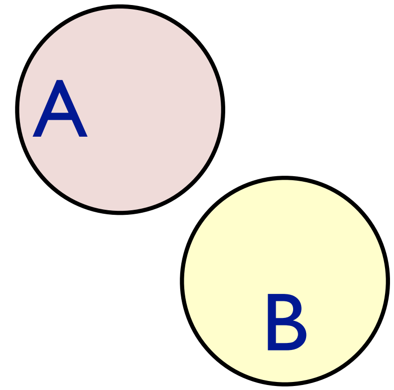
Changing the visualization often helps. Instead of tracing paths on the grid above, list choices. You walk 7 blocks; at each intersection choose N or E; must choose N exactly 3 times.

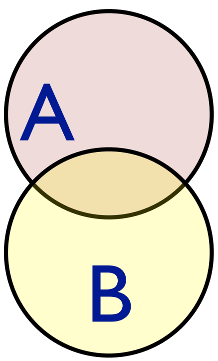
NNNEEEE
NNENEEE
NNEENEE
...
EEEENNN

another general counting rule: inclusion-exclusion

If two sets or events A and B are *disjoint*, aka *mutually exclusive*, then

$$|A \cup B| = |A| + |B|$$



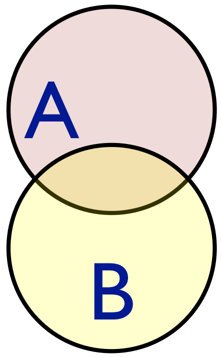


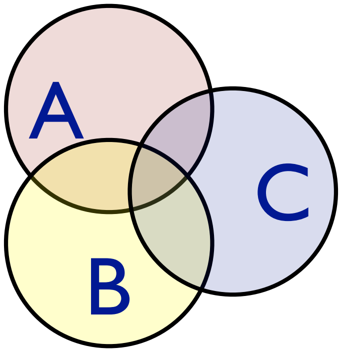
More generally, for two sets or events A and B, *whether or not they are disjoint,*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

inclusion-exclusion

(Why? Points in $A \cap B$ are *double-counted*: once in $|A|$, once in $|B|$; "$-|A \cap B|$" corrects)





$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap C| - |A \cap B| + |A \cap B \cap C|$$

General: + singles - pairs + triples - quads + ...

How many of $1, 2, \ldots, 10$ are divisible by 2, 3, and/or 5?

Let

$$E_2 = \{x \mid 1 \leq x \leq 10 \land x \text{ is a multiple of } 2\}$$

$$E_3 = \{x \mid 1 \leq x \leq 10 \land x \text{ is a multiple of } 3\}$$

$$E_5 = \{x \mid 1 \leq x \leq 10 \land x \text{ is a multiple of } 5\}$$

Notation: "AB" means "A and B"

$$\begin{aligned}
&|E_2 \cup E_3 \cup E_5| \\
&\quad = |E_2| + |E_3| + |E_5| - |E_2E_3| - |E_2E_5| - |E_3E_5| + |E_2E_3E_5| \\
&\quad = \left\lfloor \frac{10}{2} \right\rfloor + \left\lfloor \frac{10}{3} \right\rfloor + \left\lfloor \frac{10}{5} \right\rfloor - \left\lfloor \frac{10}{2 \cdot 3} \right\rfloor - \left\lfloor \frac{10}{2 \cdot 5} \right\rfloor - \left\lfloor \frac{10}{3 \cdot 5} \right\rfloor + \left\lfloor \frac{10}{2 \cdot 3 \cdot 5} \right\rfloor \\
&\quad = 5 + 3 + 2 - 1 - 1 - 0 + 0 \\
&\quad = 8
\end{aligned}$$

[Of course, the exceptions are 1 (too small) and 7 (prime) – easy to see for a concrete case like 1..10, but less obvious in general.]

more counting: the pigeonhole principle

If there are n pigeons in k holes and n > k, then some hole contains more than one pigeon.

More precisely, some hole contains at least $\lceil n/k \rceil$ pigeons.

There are two people in London who have the same number of hairs on their head.

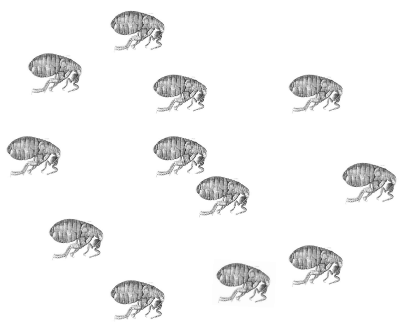
Typical head ~ 150,000 hairs

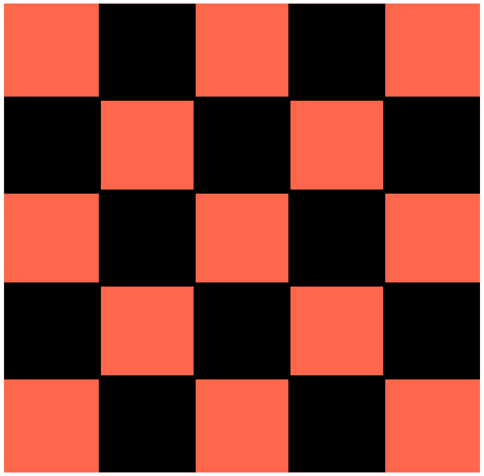
Let's say max-hairy-head ~ 1,000,000 hairs

Since there are more than 1,000,000 people in London…

Another example:

25 fleas sit on a 5 x 5 checkerboard, one per square. At the stroke of noon, all jump across an edge (not a corner) of their square to an adjacent square. Two must end up in the same square. Why?





Product Rule: n_i outcomes for A_i: $\prod_i n_i$ in total (tree diagram)

Permutations:

ordered lists of *n* objects, no repeats: $n(n-1)\ldots 1 = n!$

ordered lists of *r* objects from *n*, no repeats: $n!/(n-r)!$

Combinations:

"*n* choose *r*," aka binomial coefficients,

unordered lists of *r* objects from *n* $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Binomial Theorem: $(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$

Inclusion-Exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$

Pigeonhole Principle

Try to do everything two different ways