Now $\theta_2$:

$$\frac{\partial}{\partial\theta_2}\ln L(x_1,\cdots x_n|\theta_1,\theta_2)=\sum_{i=1}^{n}\left(-\frac{1}{2}\cdot\frac{\cancel{2\pi}}{\cancel{2\pi}\theta_2}+\frac{(x_i-\theta_1)^2}{2\theta_2^2}\right)=0$$

$$\sum_{i=1}^{n}\left(-\hat{\theta}_2+(x_i-\hat{\theta}_1)^2\right)=0$$

$$\sum_{i=1}^{n}(x_i-\hat{\theta}_1)^2=n\hat{\theta}_2$$

$$\hat{\theta}_2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\theta}_1)^2$$ ; the sample variance

$$\frac{\partial}{\partial\theta_1}\ln L(\cdots)=g(\theta_1,\theta_2)=0$$

$$\frac{\partial}{\partial\theta_2}\ln L(\cdots)=h(\theta_1,\theta_2)=0$$

$$Var(X)=E[(X-\mu)^2]$$

Check that $\theta_2$ is a maximum: the second derivative $\frac{\partial^2}{\partial\theta_2^2}\ln L(x_1,\cdots,x_n|\theta_1,\theta_2)$ is negative at the point $\theta_2=\hat{\theta}_2$.

samples from $Ber(p)$ : MLE : $\hat{p}$

" " $N(\mu,\sigma^2)$ : MLE : $\hat{\theta}_1,\hat{\theta}_2$

Bias:

Defn: An estimator $\hat{\theta}$ of $\theta$ is *unbiased* iff
$E[\hat{\theta}] = \theta$.

Note that $x_i$ are r.v.'s, so $\hat{\theta}$ is a r.v. with an expectation. It's desirable that it be $\theta$.

Ex: For estimator $\hat{\theta}_1$ of $\mu$ in $N(\mu, \sigma^2)$,

$$E[\hat{\theta}_1] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} E[x_i] = \frac{1}{n}\cdot n\mu = \mu = \theta_1 .$$

so $\hat{\theta}_1$ is an unbiased estimator.

$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\theta}_1)^2$ is a biased estimator.

$\hat{\theta}_2' = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\theta}_1)^2$ is an unbiased estimator.

For large $n$, the ratio is $\frac{n-1}{n} \to 1$.

$\hat{\theta}_2$ slightly underestimates $\theta_2 = \sigma^2$