Law of Large Numbers

Consider i.i.d. random variables $X_1, X_2, \ldots$, where $E[X_i] = \mu < \infty$ and $Var(X_i) = \sigma^2 < \infty$.

<u>Defn</u>: The <u>sample mean</u> of the first n of these is $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Each X_i is called a <u>sample</u>.

$$E[\overline{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{1}{n}\cdot n\mu = \mu.$$

$$Var(\overline{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

Recall CLT: As $n \to \infty$, $\overline{X}_n \to N\left(\mu, \frac{\sigma^2}{n}\right)$.

<u>Defn</u>: μ is the <u>population mean</u>, σ^2 is the <u>population variance</u>.

Since $E[\overline{X}_n] = \mu$ and $Var(\overline{X}_n) = \frac{\sigma^2}{n}$, as n increases, ~~the~~ $\overline{X}_n$ more likely to be close to μ.

<u>Theorem (Weak Law of Large Numbers)</u>:

For any $\varepsilon > 0$, as $n \to \infty$,

$$P(|\overline{X}_n - \mu| > \varepsilon) \to 0.$$

<u>Proof</u>: By Chebyshev's Inequality,

$$P(|\overline{X}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty.$$

<u>Strong Law of Large Numbers</u>: $P\left(\lim_{n\to\infty} \overline{X}_n = \mu\right) = 1$.

Maximum Likelihood Estimators

Given independent samples $x_1, x_2, \ldots, x_n$ from some distribution $f(x|\theta)$, estimate θ.

Ex: Given samples HHTHH of independent flips of a coin, estimate $\theta = P(\text{heads})$.

$P(x|\theta)$: Probability of event x given model θ.
Viewed as a function of x (θ fixed), it's a probability
" " " " " θ (x fixed), it's a likelihood
and often written $L(x|\theta)$.

What θ makes HHTHH most likely? I.e., what θ maximizes $L(HHTHH|\theta)$?

Approach: $\frac{\partial}{\partial\theta} L(\vec{x}|\theta) = 0$ and solve for $\hat{\theta} = \theta$.