Linearity of Expectation

<u>Defn</u>: If X is a random variable and g is a function $E[g(X)] = \sum_x g(x)p(x)$

<u>Theorem</u>: For any constants a and b,
$E[aX+b] = aE[X]+b$.

<u>Proof</u>: $E[aX+b] = \sum_x (ax+b)p(x)$

$$= \sum_x (axp(x)+bp(x)) = \sum_x axp(x) + \sum_x bp(x)$$

$$= a\sum_x xp(x) + b\sum_x p(x) = aE[X]+b$$

<u>Ex</u>: You pay \$1 to play the following game. A ~~fair~~ coin is flipped up to and including the first head, and you are paid 12¢ per flip. Do you expect to win or lose money?

with prob. 1/8 of heads

Let X be the number of flips until the first head.

Your expected gain is $E[12X-100] = 12E[X]-100$
$= 12\cdot 8 - 100 = -4$

because $E[X] = \frac{1}{1/8} = 8$

<u>Theorem</u>: Let X and Y be two random variables, possibly dependent. Then $E[X+Y] = E[X]+E[Y]$.

<u>Proof</u>: Let $X(s)$, $Y(s)$ be the values of X, Y for some $s\in\Omega$.

$$E[X+Y] = \sum_{s\in\Omega}(X(s)+Y(s))P(s) = \sum_{s\in\Omega}X(s)P(s) + \sum_{s\in\Omega}Y(s)P(s)$$

$$= E[X]+E[Y]$$

Ex: Let X be the number of heads when a coin with $P(\text{Heads}) = p$ is flipped n times independently.

Let $X_i = \begin{cases} 1, & \text{if } i^{\text{th}} \text{ flip is Heads} \\ 0, & \text{otherwise.} \end{cases}$, for $1 \le i \le n$

"Indicator random variables": only have values 0 or 1.

$$E[X_i] = 1 \cdot P(X_i = 1) + \cancel{0 \cdot P(X_i = 0)}$$
$$= P(i^{\text{th}} \text{ flip is Heads})$$
$$= p$$

$$X = \sum_{i=1}^{n} X_i$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

Linearity is special. In general,

$$E[XY] \neq E[X]E[Y]$$
$$E[X^2] \neq (E[X])^2$$
$$E[\sqrt{X}] \neq \sqrt{E[X]}$$
$$E(X!) \neq (E[X])!$$

Variance