Random Variables

Defn: a random variable is a numeric function of the outcome.

Ex: 1. Number of heads when a coin is flipped 20 times.
2. Total of 2 die rolls.
3. Number of coin flips until first head.
Suppose coin has probability p of heads. Let X be the number of independent flips up to and including the first head. X is a r.v.

$P(X=1) = P(H) = p$

$P(X=2) = P(TH) = p(1-p)$

$P(X=i) = P(T^{i-1}H) = p(1-p)^{i-1}$

Defn: If a random variable has a countable number of possible values, it is called discrete.

Defn: If X is a discrete r.v. with values in ~~some~~ countable set T, the probability mass function (pmf) of X is

$$p(a) = p_X(a) = \begin{cases} P(X=a), & \text{if } a \in T \\ 0, & \text{otherwise.} \end{cases}$$

Note: $\sum_{a \in T} p(a) = 1$

Defn: For a discrete r.v. X with pmf p, the expectation (or expected value or mean) of X is $E[X] = \sum_x x p(x)$.

In the case of equally probable outcomes, $E[X]$ is just the average value. But in general, it's a weighted average; each x weighted by its probability $p(x)$.

For the binomial random variable X of slide 13 (# of heads in n ind. coin flips)

$$E[X] = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = ?$$

Back to geometric random variable.
Let X be # of independent coin flips up to and including first head of a coin with prob p of heads.
pmf of X: $p(i) = p(1-p)^{i-1}$

$$E[X] = \sum_{i=1}^{\infty} i p(i) = \sum_{i=1}^{\infty} i p (1-p)^{i-1} = p \sum_{i=1}^{\infty} i (1-p)^{i-1}$$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r} = (1-r)^{-1}, \quad \text{if } |r|<1$$

Take $\frac{d}{dr}$ of both sides:

$$\sum_{i=0}^{\infty} i r^{i-1} = (1-r)^{-2} = \frac{1}{(1-r)^2}$$

$$E[X] = p \sum_{i=1}^{\infty} i (1-p)^{i-1} = p \cdot \frac{1}{(1-(1-p))^2} = \frac{1}{p}$$

$\underline{Ex}$: if $p = \frac{1}{2}$, $E[X] = 2$

if $p = \frac{1}{10}$, $E[X] = 10$