Random Variable (rv): A numeric function \( X: \Omega \to \mathbb{R} \) of the outcome.

Range/Support: The support/range of a random variable \( X \), denoted \( \Omega_X \), is the set of all possible values that \( X \) can take on.

Discrete Random Variable (drv): A random variable taking on a _______________ (either finite or countably infinite) number of possible values.

Probability Mass Function (pmf) for a discrete random variable \( X \): a function \( p_X: \Omega_X \to [0,1] \) with \( p_X(x) = P(X = x) \) that maps possible values of a discrete random variable to the probability of that value happening, such that \( \sum x p_X(x) = 1 \).

Expectation (expected value, mean, or average): The expectation of a discrete random variable is defined to be

\[
E[X] = \sum \quad
\]

The expectation of a function of a discrete random variable \( g(X) \) is

\[
E[g(X)] = \sum \quad
\]

Linearity of Expectation: Let \( X \) and \( Y \) be random variables, and \( a, b, c \in \mathbb{R} \). Then,

\[
E[aX + bY + c] = \quad
\]

Exercises

1. Suppose we have \( N \) items in a bag, \( K \) of which are successes. Suppose we draw (without replacement) until we have \( k \) successes, \( k \leq K \leq N \). Let \( X \) be the number of draws until the \( k^{th} \) success. What is \( \Omega_X \)? What is \( p_X(n) = P(X = n) \)? (We say \( X \) is a “negative hypergeometric” random variable).
2. A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability $p_1$, to the left with probability $p_2$, and doesn’t move with probability $p_3$, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let $X$ be the location of the frog. Find the probability mass function for $X$, $p_X(k)$. Find $E[X]$. Find the probability mass function for $Y = |X|$, $p_Y(k)$, and $E[Y]$.

3. Suppose we have $r$ independent random variables $X_1, \ldots, X_r$ that each represent the number of coins flipped up to and including the first head, where $P(\text{head}) = p$. Recall that each $X_i$ has probability mass function,

$$p_{X_i}(k) = P(X_i = k) = (1 - p)^{k-1}p$$

a) What do you think $E[X_i]$ should be (without calculations) if $p = \frac{1}{2}$? If $p = \frac{1}{3}$? In the general case? (Proof in lecture next time.)

b) Suppose we define $X = X_1 + \cdots + X_r$. What does $X$ represent, in English words? (Hint: think of performing each “trial” one after the other.)

c) What is $\Omega_X$? Find the probability mass function for $X$, $p_X(k)$.

d) Find $E[X]$ using linearity of expectation.