Equivalently, if $Var(Y) = \sigma^2$, then
$P(|Y-\mu| \geq t\sigma) \leq \frac{1}{t^2}$, for any $t>0$.

<u>Proof</u>: Let $X = (Y-\mu)^2$.

$$P(|Y-\mu| \geq \alpha) = P(X \geq \alpha^2)$$
$$\leq \frac{E[X]}{\alpha^2} \quad \text{(Markov, since } X \geq 0\text{)}$$
$$= \frac{Var(Y)}{\alpha^2}$$

<u>Ex</u>: Average daily expense is $\mu = 1500$. Suppose $\sigma = 200$. Let Y be daily expense.

$$P(Y \geq 2500) = P(Y-1500 \geq 1000)$$
$$\leq P(|Y-1500| \geq 1000) \quad \text{(either } Y \geq 2500 \text{ or } Y \leq 500\text{)}$$
$$\leq \frac{Var(Y)}{1000^2} = \frac{200^2}{1000^2} = \left(\frac{1}{5}\right)^2 = \frac{1}{25}$$

<u>Cantelli's Inequality</u> (one-sided Chebyshev):
If $\alpha > 0$, then $P(Y-\mu \geq \alpha) \leq \dfrac{Var(Y)}{Var(Y)+\alpha^2}$.

<u>Chernoff bounds</u>:

<u>Theorem</u>: Suppose $X \sim Bin(n,p)$ and $\mu = E[X] = np$. For any $0<\delta<1$,

$$P(X \geq (1+\delta)\mu) \leq e^{-\frac{1}{3}\delta^2\mu}$$
$$P(X \leq (1-\delta)\mu \leq e^{-\frac{1}{2}\delta^2\mu}$$

Law of Large Numbers

Consider i.i.d random variables $X_1, X_2, X_3, \ldots$, where $E[X_i] = \mu < \infty$ and $Var(X_i) = \sigma^2 < \infty$.

Define sample mean $M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$E[M_n] = \mu$ and
$Var(M_n) = \sigma^2/n$ (from linearity).

As n increases, M_n is more likely to be close to μ.

Theorem (Weak Law of Large Numbers):
For any $\varepsilon > 0$, as $n \to \infty$
$$P(|M_n - \mu| > \varepsilon) \to 0.$$

Proof: By Chebyshev's inequality
$$P(|M_n - \mu| > \varepsilon) \leq \frac{Var(M_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0$$
as $n \to \infty$.

Strong Law of Large Numbers:
$$P(\lim_{n\to\infty} M_n = \mu) = 1.$$

Strong $\Rightarrow$ Weak, but Weak $\not\Rightarrow$ Strong.

In some sense, these laws are a consequence of the Central Limit Theorem, which says distribution of $M_n \to N(\mu, \sigma^2/n)$ as $n \to \infty$.