Equivalently, if \( \text{Var}(Y) = \sigma^2 \), then
\[
P(|Y - \mu| \geq t \sigma) \leq \frac{1}{t^2}, \text{ for any } t > 0.
\]

**Proof:** Let \( X = (Y - \mu)^2 \).
\[
P(|Y - \mu| \geq \alpha) = P(X \geq \alpha^2)
\]
\[
\leq \frac{E[X]}{\alpha^2} \quad \text{(Markov, since } X \geq 0)\]
\[
= \frac{\text{Var}(Y)}{\alpha^2}
\]

**Ex:** Average daily expense is \( \mu = 1500 \). Suppose \( \sigma = 200 \). Let \( Y \) be daily expense.
\[
P(Y \geq 2500) = P(Y - 1500 \geq 1000)
\]
\[
\leq P(|Y - 1500| \geq 1000) \quad \text{(either } Y \geq 2500 \text{ or } Y \leq 500)\]
\[
\leq \frac{\text{Var}(Y)}{1000^2} = \frac{200^2}{1000^2} = \left( \frac{1}{5} \right)^2 = \frac{1}{25}
\]

**Cantelli's Inequality (one-sided Chebyshev):**
If \( \alpha > 0 \), then
\[
P(Y - \mu \geq \alpha) \leq \frac{\text{Var}(Y)}{\alpha^2}
\]

**Chernoff bounds:**
\[\text{Theorem: Suppose } X \sim \text{Bin}(n, p), \text{ for any } 0 < \delta < 1,\]
\[
P(X \geq (1 + \delta) \mu) \leq e^{-\frac{\delta^2 \mu}{3}}
\]
\[
P(X \leq (1 - \delta) \mu) \leq e^{-\frac{\delta^2 \mu}{2}}
\]
Law of Large Numbers

Consider i.i.d. random variables $X_1, X_2, X_3, \ldots$ where $E[X_i] = \mu < \infty$ and $Var(X_i) = \sigma^2 < \infty$. Define sample mean $M_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. 

- $E[M_n] = \mu$ and
- $Var(M_n) = \frac{\sigma^2}{n}$ (from linearity).

As $n$ increases, $M_n$ is more likely to be close to $\mu$.

**Theorem (Weak Law of Large Numbers):**

For any $\varepsilon > 0$, as $n \to \infty$

\[ P \left( |M_n - \mu| > \varepsilon \right) \to 0. \]

**Proof:** By Chebyshev's inequality,

\[ P \left( |M_n - \mu| > \varepsilon \right) \leq \frac{Var(M_n)}{\varepsilon^2} = \frac{\sigma^2}{n \varepsilon^2} \to 0 \]

as $n \to \infty$.

**Strong Law of Large Numbers:**

\[ P \left( \lim_{n \to \infty} M_n = \mu \right) = 1. \]

Strong $\rightarrow$ Weak, but Weak $\not\rightarrow$ Strong.

In some sense, these laws are a consequence of the Central Limit Theorem, which says (distribution of $M_n \to N(\mu, \sigma^2/n)$ as $n \to \infty$).