$E[aX+b] = aE[X]+b$

$E[X+Y] = E[X]+E[Y]$

$E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)\,dx$

$Var(X) = E[(X-\mu)^2] = E[X^2]-(E[X])^2$

where $\mu = E[X]$.

X and Y are independent iff

$\forall A \forall B\ P(X\in A \cap Y \in B) = P(X\in A)P(Y\in B)$.

(Discrete: X and Y ind iff $\forall x \forall y\ P(X=x \cap Y=y) = P(X=x)P(Y=y)$)

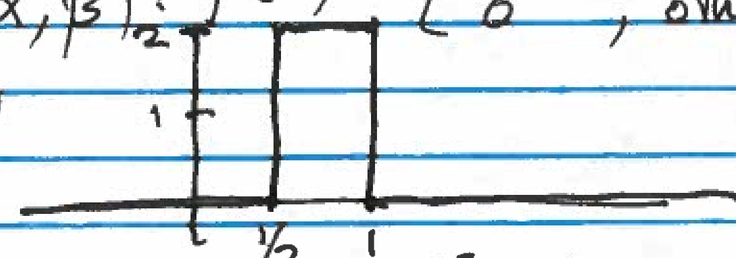
If X and Y are ind, $Var(X+Y) = Var(X)+Var(Y)$

Special continuous random variables

1. Uniform:

$X \sim Uni(\alpha,\beta)$: $f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } x\in[\alpha,\beta] \\ 0, & \text{otherwise} \end{cases}$

Ex $Uni(\frac{1}{2},1)$



$$E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_\alpha^\beta \frac{x}{\beta-\alpha}\,dx$$

$$= \frac{1}{\beta-\alpha}\int_\alpha^\beta x\,dx = \frac{1}{2(\beta-\alpha)}\cdot x^2\Big|_\alpha^\beta = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)}$$

$$= \frac{1}{2}(\alpha+\beta).$$

$$E[X^2] = \int_\alpha^\beta x^2 \cdot \frac{1}{\beta-\alpha}\,dx = \frac{1}{\beta-\alpha}\cdot\frac{1}{3}x^3\Big|_\alpha^\beta$$

$$\frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{1}{3}(\alpha^2+\alpha\beta+\beta^2)$$

$$Var(X) = E[X^2] - (E[X])^2 =$$

$$= \frac{1}{3}(\alpha^2+\alpha\beta+\beta^2) - \frac{1}{4}(\alpha^2+2\alpha\beta+\beta^2)$$

$$= \frac{1}{12}\alpha^2 - \frac{1}{6}\alpha\beta + \frac{1}{12}\beta^2 = \frac{1}{12}(\alpha-\beta)^2 = \frac{1}{12}(\beta-\alpha)^2$$

2. Exponential distribution: random, independent events happen at an average rate of λ per time unit. An exponential random variable is the time until the next event.

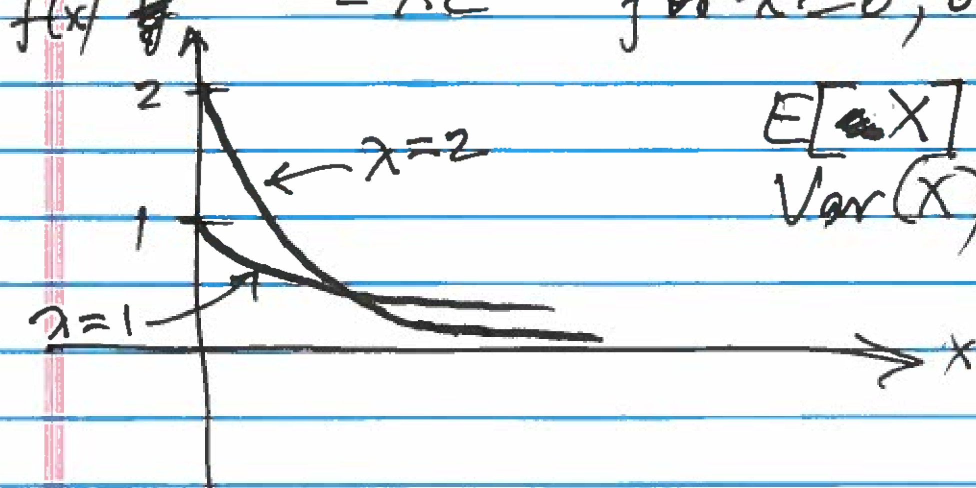
Ex: time until next α particle emitted.
time until next packet arrives at a server.

$X \sim Exp(\lambda)$

$P(X>t) = e^{-\lambda t}$

$F(t) = P(X<t) = 1-e^{-\lambda t}$ for $t \geq 0$, else 0

$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}(1-e^{-\lambda x}) = (-e^{-\lambda x})(-\lambda)$
$= \lambda e^{-\lambda x}$ for $x \geq 0$, otherwise $f(x)=0$



$E[X] = \frac{1}{\lambda}$

$Var(X) = \frac{1}{\lambda^2}$

Exponential is the continuous analog of the geometric distribution: both measure "time" until next event.

Poisson is number of events in a particular unit of time.
Exponential is time to the next event.

Memorylessness property of $X \sim Exp(\lambda)$:

$$P(X > s+t \mid X > s) = P(X > t), \text{ for any } s,t > 0.$$