Theorem: $Var(X) = E[X^2] - (E[X])^2$

Proof: Let $\mu = E[X]$.

$$Var(X) = E[(X-\mu)^2] = \sum_x (x-\mu)^2 p(x)$$
$$= \sum_x (x^2 - 2\mu x + \mu^2) p(x)$$
$$= \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x)$$
$$= E[X^2] - 2\mu E[X] + \mu^2 \cdot 1$$
$$= E[X^2] - 2\mu^2 + \mu^2$$
$$= E[X^2] - (E[X])^2$$

Theorem: $Var(aX+b) = a^2 Var(X)$, for any constants a and b.

Proof: ~~$Var(aX+b)$~~ Let $\mu = E[X]$.

$$Var(aX+b) = E[((aX+b) - (a\mu+b))^2]$$
$$= E[(a(X-\mu))^2]$$
$$= E[a^2 (X-\mu)^2]$$
$$= a^2 E[(X-\mu)^2]$$
$$= a^2 Var(X)$$

Consequence is that, in general, $Var(X+Y) \neq Var(X) + Var(Y)$. In particular

$$Var(X+X) = Var(2X) = 4\,Var(X) \neq Var(X) + Var(X)$$

Defn: Random variables X and Y are <u>independent</u> iff
$$\forall x \forall y \; P(X=x \cap Y=y) = P(X=x)\,P(Y=y).$$

Ex: Flip a fair coin $2n$ times independently. Let
X be the number of heads in the first n flips,
Y .. .. last n flips,
Z .. .. all $2n$ flips.

$$\underbrace{\cdots\cdots\cdots}_{X \text{ heads}}\underbrace{-\,-\,-\,-\,-}_{Y \text{ heads}} \qquad Z = X+Y.$$

X and Y are independent: follows from defn.
X and Z are dependent:
$P(X=0) > 0$
$P(Z=n+1) > 0$
But $0 = P(X=0 \cap Z=n+1) \neq P(X=0)P(Z=n+1) > 0$

Theorem: If X and Y are independent, then
$E[XY] = E[X]\,E[Y]$.

Proof:
$$E[XY] = \sum_x \sum_y xy\,P(X=x \cap Y=y)$$
$$= \sum_x \sum_y xy\,P(X=x)P(Y=y) \quad (\text{ind.})$$
$$= \left(\sum_x x P(X=x)\right)\left(\sum_y y P(Y=y)\right)$$
$$= E[X]\,E[Y]$$

<u>Theorem</u>: If X and Y are independent r.v.'s, $Var(X+Y) = Var(X) + Var(Y)$.

<u>Proof</u>:

$$
\begin{aligned}
Var(X+Y) &= E[(X+Y)^2] - (E[X+Y])^2 \\
&= E[X^2+2XY+Y^2] - (E[X]+E[Y])^2 \\
&= E[X^2]+E[2XY]+E[Y^2] \\
&\quad - \left((E[X])^2+2E[X]E[Y]+(E[Y])^2\right) \\
&= E[X^2]-(E[X])^2+E[Y^2]-(E[Y])^2 \\
&\quad +2E[X]E[Y]-2E[X]E[Y] \qquad \text{(ind)} \\
&= Var(X)+Var(Y)
\end{aligned}
$$

Recall: $Var(X+X) \neq Var(X)+Var(X)$

A collection of special and important distributions (i.e., random variables).

A.