Theorem: \( \text{Var}(X) = E[X^2] - (E[X])^2 \)

Proof: Let \( \mu = E[X] \).

\[
\text{Var}(X) = E[(X-\mu)^2] = \sum_x (x-\mu)^2 p(x) \\
= \sum_x (x^2 - 2\mu x + \mu^2) p(x) \\
= \sum_x x^2 p(x) - 2\mu \sum_x xp(x) + \mu^2 \sum_x p(x) \\
= E[X^2] - 2\mu E[X] + \mu^2 \\
= E[X^2] - 2\mu^2 + \mu^2 \\
= E[X^2] - (E[X])^2
\]

Theorem: \( \text{Var}(aX+b) = a^2 \text{Var}(X) \), for any constants \( a \) and \( b \).

Proof: Let \( \mu = E[X] \).

\[
\text{Var}(aX+b) = E[(aX+b) - (a\mu+b)]^2 \\
= E[a(X-\mu)]^2 \\
= E[a^2 (X-\mu)^2] \\
= a^2 E[(X-\mu)^2] \\
= a^2 \text{Var}(X)
\]

Consequence is that, in general, \( \text{Var}(X+Y) \neq \text{Var}(X) + \text{Var}(Y) \). In particular, \( \text{Var}(X+X) = \text{Var}(2X) = 4 \text{Var}(X) \neq \text{Var}(X) + \text{Var}(X) \).
Defn: Random variables $X$ and $Y$ are independent if
$$
\forall x,y \quad P(X=x \cap Y=y) = P(X=x)P(Y=y).
$$

Ex: Flip a fair coin $2n$ times. Let $X$ be the number of heads in the first $n$ flips, $Y$ the number of heads in the last $n$ flips, $Z$ the number of heads in all $2n$ flips.

$Z = X + Y$.

$X$ and $Y$ are independent; follows from defn.

$X$ and $Z$ are dependent:

$P(X=0) > 0$

$P(Z=n+1) > 0$

But $0 = P(X=0 \cap Z=n+1) \neq P(X=0)P(Z=n+1) > 0$

**Theorem:** If $X$ and $Y$ are independent, then
$$
E[XY] = E[X]E[Y].
$$

**Proof:**

$$
E[XY] = \sum_x \sum_y xy P(X=x \cap Y=y)
$$

$$
= \sum_x \sum_y xy P(X=x) P(Y=y) \quad \text{(ind.)}
$$

$$
= \left(\sum_x P(X=x)\right) \left(\sum_y P(Y=y)\right)
$$

$$
= E[X]E[Y]
$$
Theorem: If $X$ and $Y$ are independent r.v.'s, then
$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y).$$

Proof:

$$\text{Var}(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$$
$$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$$
$$= E[X^2] - (E[X])^2 + E[Y^2] - (E[Y])^2$$
$$= \text{Var}(X) + \text{Var}(Y)$$

Recall: \( \text{Var}(X+X) \neq \text{Var}(X) + \text{Var}(X) \)

A collection of special and important distributions (i.e., random variables).