A simple example of linearity

Shuffle 4 aces, put 2 aside and let the other 2 be card 1 and card 2 on the table. Let X be the number of spades on the table. Compute $E[X]$.

Let $X_1 = \begin{cases} 1, & \text{if card 1 is } \spadesuit A \\ 0, & \text{otherwise} \end{cases}$

$X_2 = \begin{cases} 1, & \text{if card 2 is } \spadesuit A \\ 0, & \text{otherwise} \end{cases}$

$X = X_1 + X_2$

$$E[X_1] = 1 \cdot P(X_1 = 1) + 0 \cdot P(X_1 = 0)$$
$$= \boxed{P(\text{card 1 is } \spadesuit A) = 1/4}$$
$$E[X_2] = \boxed{1 \cdot P(X_2 = 1) + 0 \cdot P(X_2 = 0)}$$
$$= \boxed{P(\text{card 2 is } \spadesuit A) = 1/4}$$
$$E[X] = E[X_1 + X_2] = E[X_1] + E[X_2] \quad (\text{lin of exp})$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Let E_1 = "card 1 is $\spadesuit A$"

E_2 = "card 2 is $\spadesuit A$"

E_1 and E_2 are dependent.

$$P(E_2) = P(E_2 \mid E_1) P(E_1) + P(E_2 \mid \bar{E}_1) P(\bar{E}_1)$$
$$= 0 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

Variance. Consider 2 fair coin games between A and B.

1. A gain per flip is $X = \begin{cases} +1, & \text{if heads} \\ -1, & \text{if tails} \end{cases}$
2. A gain per flip is $Y = \begin{cases} +1000, & \text{if heads} \\ -1000, & \text{if tails.} \end{cases}$

$E[X] = E[Y] = 0$.

~~How~~ do we measure the variability of X and Y.

Defn: Let X be a r.v. with $E[X] = \mu$. The variance of X is

$$Var(X) = E[(X-\mu)^2]$$ and often denoted σ^2.

Defn: Standard deviation of X is $\sigma = \sqrt{Var(X)}$.

Continuing the coin game example:

$$Var(X) = E[(X-\mu)^2] = \cancel{E[X^2]}$$
$$\cancel{= 1^2 \cdot P(X=1) + 0^2 \cdot P(X=0)} =$$
$$= E[X^2]$$
$$= 1^2 \cdot P(X=1) + (-1)^2 \cdot P(X=-1).$$
$$= 1 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{2} = 1$$

$$Var(Y) = E[(Y-\mu)^2] = E[Y^2]$$
$$= 1000^2 P(Y=1000) + (-1000)^2 P(Y=-1000)$$
$$= 1{,}000{,}000$$

Examples of standard deviation from Slide Pack 6, slides 33 and 35.