Defn: $E[g(X)] = \sum_x g(x)p(x)$.

Linearity of expectation

Theorem: For any constants a and b,
$E[aX+b] = aE[X]+b$.

Proof: $E[aX+b] = \sum_x (ax+b)p(x) = \sum_x axp(x) + \sum_x bp(x)$
$= a\sum_x xp(x) + b\sum_x p(x) = aE[X]+b$

Ex. At a casino, you pay \$100 to play the following game: they flip a coin with prob. $p = \frac{1}{8}$ of heads, and pay you \$12 for each flip up to and including the first head. Do you expect to win, lose or break even?

Let X be the number of flips. Your gain is $12X - 100$.

$E[12X-100] = 12E[X]-100 = 12\cdot 8 - 100 = -4$

Theorem: Let X and Y be two random variables, possibly dependent. Then
$E[X+Y] = E[X] + E[Y]$.

Proof: Let $X(s)$ and $Y(s)$ be the values of X and Y on outcome $s \in \Omega$.

$E[X+Y] = \sum_{s\in\Omega} (X(s)+Y(s))P(s) =$

$\sum_{s\in\Omega} X(s)P(s) + \sum_{s\in\Omega} Y(s)P(s) = E[X]+E[Y]$

Ex: Let X be the number of heads when a coin with prob p of heads is flipped n times. What is $E[X]$?

Define an indicator random variable

$$X_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

for each $i \in \{1, 2, \cdots, n\}$.

$$E[X_i] = 1 \cdot P(X_i = 1) + \cancel{0 \cdot P(X_i = 0)}$$

$$= P(X_i = 1) = p.$$

$$X = \sum_{i=1}^{n} X_i$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] \quad \text{(lin. of exp.)}$$

$$= \sum_{i=1}^{n} p = pn$$

$$E[aX + bY + c] = E[aX + bY] + c$$

$$= E[aX] + E[bY] + c$$

$$= a\,E[X] + b\,E[Y] + c$$

Note: Linearity is special!

$$E[XY] \neq E[X]E[Y]$$

$$E[X^2] \neq (E[X])^2$$

$$E[X!] \neq (E[X])!$$

$$\vdots$$