Proof: $\Rightarrow$: E, F independent $\Rightarrow$

$$P(E)P(F) = P(E\cap F) = P(E|F)P(F)$$

Divide both sides by $P(F)$: $P(E) = P(E|F)$.

$\Leftarrow$: $P(E|F) = P(E) \Rightarrow P(E\cap F) = P(E|F)P(F)$

$$= P(E)P(F)$$

$\Rightarrow E, F$ independent.

Suppose a biased coin comes up heads with probability p. Suppose it is flipped n times independently.

$P(n \text{ heads}) = p^n$ (independence)

$P(\text{first } k \text{ heads and last } n-k \text{ tails}) = p^k(1-p)^{n-k}$

$P(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$

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Note:

$$1 = P(0 \text{ heads or } 1 \text{ head or } 2 \text{ heads or } \ldots \text{ or } n \text{ heads})$$

$$= \sum_{k=0}^{n} \binom{n}{k} (p)^k (1-p)^{n-k} = (p + (1-p))^n$$

(by the Binomial Theorem) $= 1^n$

$$P(\geq k \text{ heads}) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

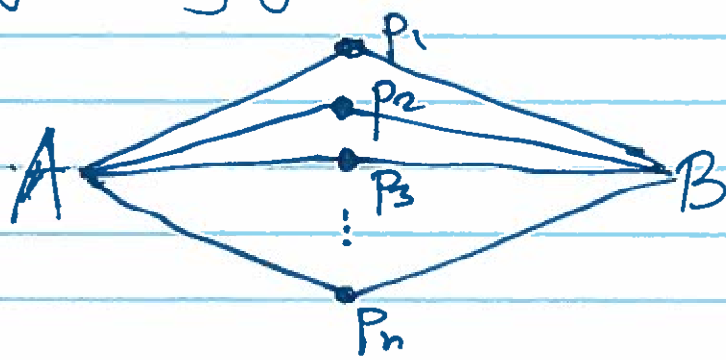
$k=2$: $P(\geq 2 \text{ heads}) = 1 - P(\leq 1 \text{ head})$

$$= 1 - \binom{n}{0} p^0 (1-p)^n - \binom{n}{1} p^1 (1-p)^{n-1}$$

$$= 1 - (1-p)^n - np(1-p)^{n-1}.$$

Network failures

1. Suppose we have n routers in parallel, where the i^{th} router fails with probability p_i, independently of the others.



P(A can communicate with B)
$= 1 - P$(all n routers fail)
$= 1 - p_1 p_2 p_3 \cdots p_n$

2. Suppose ~~to~~ the n routers are in series:

P(A can communicate with B)
$= P$(no router fails)
$= (1-p_1)(1-p_2)(1-p_3)\cdots(1-p_n).$

Random Variables

Defn: A random variable is any numeric function of the outcome.

Ex: # heads when 20 coins are flipped
total of 2 die rolls
flips until the first ~~head~~ head.
Suppose the coin has probability p of heads.
Let X be the number of coin flips up to and including the first head.

$p(1) = P(X=1) = p$
$p(2) = P(X=2) = (1-p)p$
$p(k) = P(X=k) = (1-p)^{k-1}p$

Defn: If the random variable has a countable number of possible values, it is called discrete.
If uncountable, it is called continuous.

Defn: If X is a discrete random variable with values from a countable set T, the probability mass function (PMF) of X is

$$p(a) = \begin{cases} P(X=a), & \text{if } a \in T \\ 0, & \text{otherwise.} \end{cases}$$

Note: $\sum_{a \in T} p(a) = 1$.