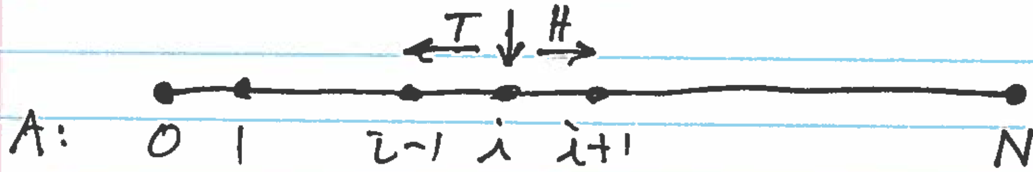
Gambler's Ruin

A has \$$i$ and B has \$$(N-i)$.
Flip a fair coin: $H \to$ A wins \$1 from B,
$T \to$ B wins \$1 from A.
Whoever first has \$$N$ wins.
Random walk on line:



Let E_i = A wins starting with \$$i$.
Condition on first flip using Law of Total Probability:

$$p_i = P(E_i) = P(E_i|H)P(H) + P(E_i|T)P(T)$$
$$= \tfrac{1}{2}(p_{i+1} + p_{i-1})$$
$$2p_i = p_{i+1} + p_{i-1}$$
$$p_i - p_{i-1} = p_{i+1} - p_i$$
$$p_2 - p_1 = p_1 - p_0 = p_1 - 0 = p_1$$
$$p_2 = 2p_1$$
$$p_i = ip_1$$
$$1 = p_N = Np_1$$
$$p_1 = 1/N$$
$$p_i = i/N$$

Isn't $1 - p_1 = \underset{1/2}{P(T)}$?
$\Rightarrow p_1 = 1/2$ but $p_1 = 1/N$
So, no.

Bayes' Theorem (Rev. Thomas Bayes, c. 1701–1761)

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}.$$

Proof:
$$P(F|E) = \frac{P(E\cap F)}{P(E)} \quad \text{(defn of cond. prob.)}$$
$$= \frac{P(E|F)P(F)}{P(E)} \quad \text{(chain rule)}$$

Bayes' Theorem reverses the conditioning.

Ex: F = patient has disease
E = patient has positive result in diagnostic test.

Corollary:
$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|\bar{F})P(\bar{F})}$$

Ex.: 60% of email is spam
90% of spam has forged header
20% of nonspam has forged header.
Let F = forged header
J = spam
What is $P(J|F)$

$$P(J|F) = \frac{P(F|J)P(J)}{P(F|J)P(J) + P(F|\bar{J})P(\bar{J})}$$
$$= \frac{0.9\times 0.6}{0.9\times 0.6 + 0.2\,(1-0.6)} \approx 0.871$$

prior prob.