Ex: Assume ♣J is face-up. Let Ω be all ways of dealing 2 Schnapsen hands. Let
Y = you are dealt 0 trumps and
Θ = opponent is " " "

$$P(\Theta|Y) = \frac{|\Theta \cap Y|}{|Y|} = \frac{\cancel{\binom{15}{5}}\binom{10}{5}}{\cancel{\binom{15}{5}}\binom{14}{5}} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{14\cdot 13\cdot 12\cdot 11\cdot 10}$$
≈ 0.126

Compare to $P(Y) \approx 0.258 \approx \binom{15}{5}/\binom{19}{5}$

Notice that restricting the sample space to Y would yield $\frac{\binom{10}{5}}{\binom{14}{5}}$.

General case: outcomes may not be equally likely.

Ex: Flip a fair coin twice. Let $\Omega = \{0, 1, 2\}$, where we are considering the number of heads in 2 flips.
$P(0) = P(2) = 1/4$ but $P(1) = 1/2$.

General defn of conditional probability:
$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

Let $F = \geq 1$ head and $E = 2$ heads.
$$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{P(E)}{P(F)} = \frac{1/4}{1-1/4} = \frac{1}{3}$$

Chain rule: $P(E \cap F) = P(E|F)P(F)$.

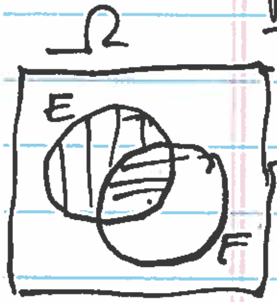
Generalize: $P(E_1 \cap E_2 \cap \cdots \cap E_n) =$
$$P(E_1)P(E_2|E_1)P(E_3|E_1,E_2)\cdots P(E_n|E_1,E_2\cdots E_{n-1})$$

Follows from chain rule by choosing
$E = E_n$, $F = E_1 \cap E_2 \cap \cdots \cap E_{n-1}$, and then recursing, i.e. by induction on n.

<u>Law of Total Probability</u>: If E and F are events,

$$P(E) = P(E|F)P(F) + P(E|\overline{F})P(\overline{F}).$$

<u>Proof</u>:



$$P(E) = P((E \cap F) \cup (E \cap \overline{F}))$$
$$= P(E \cap F) + P(E \cap \overline{F}) \quad \text{(axiom 3)}$$
$$= P(E|F)P(F) + P(E|\overline{F})P(\overline{F})$$
$$\left[\; = P(E|F)P(F) + P(E|\overline{F})(1 - P(F)) \;\right]$$

<u>Ex</u>: Sally will take either Phys or Chem. She will get an A in Phys with prob. 3/4 and an A in Chem with prob. 3/5. She flips a fair coin to decide. What is her prob. of an A?

Let A = event that she gets an A,
Phys = " " " takes physics,
Chem = " " " " chemistry.

$$P(A) = P(A|\text{Phys})P(\text{Phys}) + P(A|\text{Chem})P(\text{Chem})$$
$$= \tfrac{3}{4} \cdot \tfrac{1}{2} + \tfrac{3}{5} \cdot \tfrac{1}{2}$$
$$= 27/40$$

Generalized Law of Total Probability:
If $F_1, F_2, \ldots, F_n$ partition Ω, then

$$P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i).$$

The ordinary is $n=2$, $F_1 = F$, $F_2 = \overline{F}$.