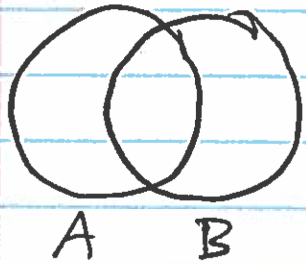
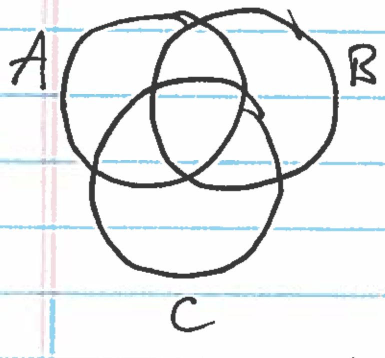
Inclusion-Exclusion Method



$$|A\cup B| = |A| + |B| - |A\cap B|.$$



$$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

In general: + singles − pairs + triples − quads ....

Ex: How many 5-card Schnapsen hands contain ≥ 1 trump, where ♣J is the trump face-up?

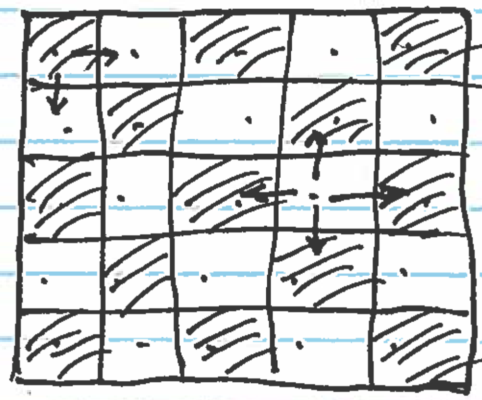
A = hands with ♣Q
B = " " ♣K
C = " " ♣T
D = " " ♣A

$$|A\cup B\cup C\cup D| = 4\binom{18}{4} - \binom{4}{2}\binom{17}{3} + \binom{4}{3}\binom{16}{2} - \binom{4}{4}\binom{15}{1}$$

$$= 12{,}240 - 4080 + 480 - 15 = 8625$$

<u>Pigeonhole Principle</u>: If n pigeons are in $(n-1)$ pigeonholes, then some pigeonhole has ≥ 2 pigeons.

<u>Ex</u>:



5x5 chessboard, one flea on each square. When a bell is rung, every flea jumps to an adjacent square. Some square now has ≥ 2 fleas.

pigeons = fleas starting on black squares (13).
pigeonholes = white squares (12).

<u>Intro to Probability</u>:

1. Sample space: set Ω of possible "outcomes" of an experiment.

<u>Ex</u>: Coin flip: $\Omega = \{H, T\}$.
2 coin flips: $\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$
Die roll: $\Omega = \{1,2,3,4,5,6\}$
#emails in a year: $\Omega = \mathbb{N} = \{0,1,2,\dots\}$
unordered 5-card Schnapsen hands, with ♣J showing:
$|\Omega| = \binom{19}{5}$.

2. Event: any $E \subseteq \Omega$.

<u>Ex</u>: ≥ 1 head in 2 coin flips: $E = \{(H,H), (H,T), (T,H)\}$
odd roll of a die: $E = \{1,3,5\}$
5-card Schnapsen hand with 0 trumps: $|E| = \binom{15}{5}$

Defn: E and F are *mutually exclusive* iff $E \cap F = \emptyset$.

3. Axioms of probability:
There is a function $\mathbb{P}$ that assigns a real number $P(E)$ to every event E satisfying:
(1) $P(E) \geq 0$, and
(2) $P(\Omega) = 1$, and
(3) If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$.

4. Consequences of the axioms:
(a) $P(\bar{E}) = P(\Omega - E) = 1 - P(E)$, because $1 = P(\Omega) = P(E \cup \bar{E}) = P(E) + P(\bar{E})$.
(b) If $E \subseteq F$, then $P(E) \leq P(F)$, because
$$P(F) = P(E \cup (F-E)) = P(E) + P(F-E) \geq P(E) + 0 = P(E)$$
(c) $P(E) \leq 1$, because of (b) and $E \subseteq \Omega$.
(d) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$: inclusion-exclusion

---

Equally likely outcomes, with Ω finite: if $a \in \Omega$, then $P(a) = 1/|\Omega|$.
Ex: Fair die roll.