How many subsets of size $k$ does an $n$-element set have? $\binom{n}{k}$.

Ex: Pike

<table>
<thead>
<tr>
<th></th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Union</td>
<td></td>
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<tr>
<td>University</td>
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<tr>
<td>Seneca</td>
<td></td>
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<tr>
<td>Work</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

lunch

$N$

How many routes from Seneca & 1st to Pike & 5th, going either Nor E at each intersection?

Same as the number of 3-character strings over $\{E, N, E^2\}$ containing 2 N's.

$$\frac{7!}{3!} = \frac{7!}{4!}$$

Identity: $\binom{n}{r} = \binom{n}{n-r}$ for any $0 \leq r \leq n$.

Identity: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, for any $1 \leq r \leq n-1$.

Choosing $r$ elements from $\{x_1, x_2, \ldots, x_n\}$

Either $x_1$ gets chosen $\binom{n-1}{r-1}$ or $x_1$ doesn't $\binom{n-1}{r}$
Binomial Theorem: \((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\),
for any nonnegative integer \(n\).

Ex: \((x+y)^3 = (x+y)(x+y)(x+y) = (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\).

"Proof": \((x+y)^n = (x+y)(x+y) \cdots (x+y)\).
From each of these \(n\) factors, choose either \(x\) or \(y\), multiply, and collect all the terms of the form \(x^k y^{n-k}\). In how many ways can we choose \(x\) from exactly \(k\) of the factors?

Corollary: \(\sum_{k=0}^{n} \binom{n}{k} = 2^n\).

Proof: \(2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1 \cdot 1^n \quad (BHT) = \sum_{k=0}^{n} \binom{n}{k}\).

Both sides of the corollary count the number of subsets of an \(n\)-element set.

Ex: \(\{1, 2, 3\}\)
- \(\emptyset , \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\): \(\binom{3}{1}\) subsets of size 1

\(\emptyset , \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\): \(\binom{3}{2}\)
Complementing.

Ex: Assume 8J is the face-up trump. How many Schnapsen starting hands contain $\geq 1$ trump?

\[
\begin{align*}
\frac{24}{1} & \cdot \frac{18}{4} = 12,240 \\
\text{8Q 8KQKQJ} & \text{ \{} \text{overcounted} \\
\text{8K 8QKQJ} & \text{ \{} \text{overcounted}
\end{align*}
\]

Ex: How many contain 0 trumps? \(\binom{15}{5}\)

How many contain $\geq 1$ trump? \(\binom{5}{5} - \binom{5}{5} = 8625\)