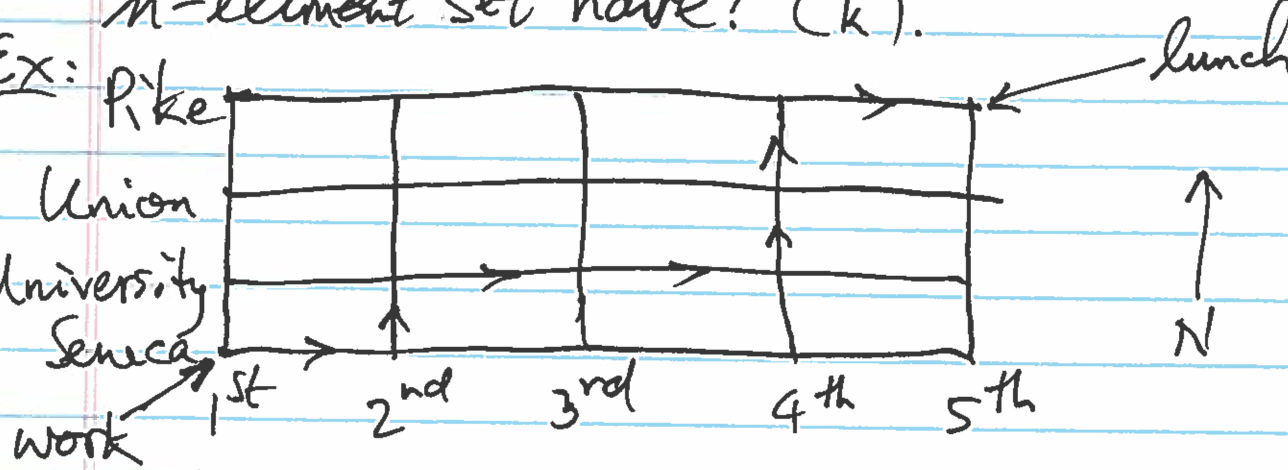
How many subsets of size k does an n-element set have? $\binom{n}{k}$.

Ex:



How many routes from Seneca & 1st to Pike & 5th, going either N or E at each intersection?

Same as the number of 7-character strings over $\{N, E\}$ containing 3 N's. $\binom{7}{3} = \frac{7!}{4!3!}$

$\underline{E}\ \underline{N}\ \underline{E}\ \underline{E}\ \underline{N}\ \underline{N}\ \underline{E}$

$\binom{7}{3} = \binom{7}{4}$

Identity: $\binom{n}{r} = \binom{n}{n-r}$ for any $0 \le r \le n$.

Identity: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, for any $1 \le r \le n-1$.

Choosing r elements from $\{x_1, x_2, \dots, x_n\}$

Either x_1 gets chosen $\binom{n-1}{r-1}$ or

x_1 doesn't $\binom{n-1}{r}$

Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, for any nonnegative integer n.

Ex: $(x+y)^3 = (x+y)(x+y)(x+y)$
$= xxx + xxy + xyx + yxx + xyy + yxy + yyx + yyy$
$= x^3 + 3x^2y + 3xy^2 + y^3$

"Proof": $(x+y)^n = \overbrace{(x+y)(x+y)\cdots(x+y)}^{n \text{ factors}}$.
From each of these n factors, choose either x or y, multiply, and collect all the terms of the form $x^k y^{n-k}$. In how many ways can we choose x from exactly k of the factors? $\binom{n}{k}$.

Corollary: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
Proof: $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k}$ (BinTh)
$= \sum_{k=0}^{n} \binom{n}{k}$

Both sides of the corollary count the number of subsets of an n-element set.

Ex: $\{1,2,3\}$
$\{\{1\}, \{2\}, \{3\}\}$: $\binom{3}{1}$ ~~subset~~ subsets of size 1
$\{\{1,2\}, \{1,3\}, \{2,3\}\}$: $\binom{3}{2}$ " " " 2

Complementing.

Ex: Assume ♣J is the face-up trump. How many Schnapsen starting hands contain ≥ 1 trump?

$$\binom{4}{1}\binom{18}{4} = 12,240$$

$$\left.\begin{matrix} ♣Q & ♣K\,♡KQJ \\ ♣K & ♣Q\,♡KQJ \end{matrix}\right\} \text{overcounted}$$

Ex: How many contain 0 trumps? $\binom{15}{5}$

How many contain ≥ 1 trump? $\binom{19}{5} - \binom{15}{5} = 8625$