Conditional Probability

P( | )

conditional probability

Conditional probability of E given F: probability that E occurs *given* that F has occurred.

"Conditioning on F"

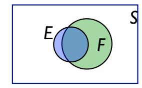
Written as P(E|F)

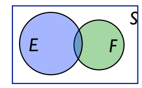
Means "P(E, given F observed)"

Sample space S reduced to those elements consistent with F (i.e. $S \cap F$)

Event space E reduced to those elements consistent with F (i.e. $E \cap F$)

With equally likely outcomes,





$$P(E \mid F) = \frac{\#\text{ of outcomes in } E \text{ consistent with } F}{\#\text{ of outcomes in } S \text{ consistent with } F} = \frac{|EF|}{|SF|} = \frac{|EF|}{|F|}$$

$$P(E \mid F) = \frac{|EF|}{|F|} = \frac{|EF|/|S|}{|F|/|S|} = \frac{P(EF)}{P(F)}$$

coin flipping

Suppose you flip two coins & all outcomes are equally likely.

What is the probability that both flips land on heads if…

• The first flip lands on heads?

Let B = {HH} and F = {HH, HT}

P(B|F) = P(BF)/P(F) = P({HH})/P({HH, HT})

= (1/4)/(2/4) = 1/2

• At least one of the two flips lands on heads?

Let A = {HH, HT, TH}, BA = {HH}

P(B|A) = |BA|/|A| = 1/3

• At least one of the two flips lands on tails?

Let G = {TH, HT, TT}

P(B|G) = P(BG)/P(G) = P(∅)/P(G) = 0/P(G) = 0

Exampls

2 random cards are selected from a deck of cards:

- What is the probability that both cards are aces given that one of the cards is the ace of spades?
- What is the probability that both cards are aces given that at least one of the cards is an ace?

conditional probability: the chain rule

General defn: $P(E \mid F) = \frac{P(EF)}{P(F)}$ where P(F) > 0

Holds even when outcomes are *not* equally likely.

What if P(F) = 0?

P(E|F) undefined: (you can't observe the impossible)

For equally likely outcomes:

$$P(E \mid F) = \frac{|EF|}{|F|} = \frac{|EF|/|S|}{|F|/|S|} = \frac{P(EF)}{P(F)}$$

Conditional Probability

Satisfies usual axioms of probability

Example:

Pr(E | F) = 1- Pr (E^c | F)

Conditional Probabilities yield a probability space

Suppose that $(S, Pr(\cdot))$ is a probability space.

Then $(S, Pr(\cdot|F))$ is a probability space for $F \subset S$ with $Pr(F) > 0$

$$0 \leq Pr(w|F) \leq 1$$

$$\sum_{w \in S} Pr(w|S) = 1$$

$E_1, E_2, \ldots, E_n$ disjoint implies

$$Pr(\cup_{i=1}^{n} E_i|F) = \sum_{i=1}^{n} Pr(E_i|F)$$

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conditional probability: the chain rule

General defn: $P(E \mid F) = \frac{P(EF)}{P(F)}$ where P(F) > 0

Implies: P(EF) = P(E|F) P(F) (*"the chain rule"*)

General definition of Chain Rule:

$$P(E_1 E_2 \cdots E_n) = P(E_1) P(E_2 \mid E_1) P(E_3 \mid E_1, E_2) \cdots P(E_n \mid E_1, E_2, \ldots, E_{n-1})$$

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Chain rule application

Draw 2 balls at random without replacement from an urn with 8 red balls and 4 white ones. What is the probability that both balls are red?

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Chain rule example

Alice and Bob play a game as follows: A die is thrown, and each time it is thrown, regardless of the history, it is equally likely to show any of the six numbers.

If it shows 5, Alice wins.

If it shows 1, 2 or 6, Bob wins.

Otherwise, they play a second round and so on.

What is P(Alice wins on n^{th} round)?

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Keys

I have n keys, one of which opens a locked door. Trying keys at random without replacement, what is the chance of opening the door on the k^{th} try?

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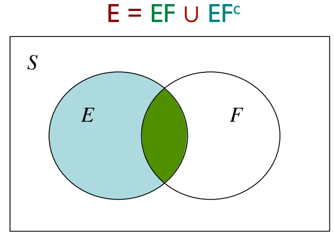
law of total probability

E and F are events in the sample space S

$E = EF \cup EF^c$



$EF \cap EF^c = \varnothing$

$\Rightarrow P(E) = P(EF) + P(EF^c)$

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law of total probability

$P(E) = P(EF) + P(EF^c)$
$\quad = P(E|F)\,P(F) + P(E|F^c)\,P(F^c)$
$\quad = P(E|F)\,P(F) + P(E|F^c)\,(1-P(F))$

weighted average, conditioned on event F happening or not.

More generally, if $F_1, F_2, \ldots, F_n$ partition S (mutually exclusive, $\bigcup_i F_i = S$, $P(F_i)>0$), then

$P(E) = \sum_i P(E|F_i)\,P(F_i)$

weighted average, conditioned on events F_i happening or not.

(Analogous to reasoning by cases; both are very handy.)

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total probability

Sally has 1 elective left to take: either Phys or Chem. She will get A with probability 3/4 in Phys, with prob 3/5 in Chem. She flips a coin to decide which to take.

What is the probability that she gets an A?

P(A) = P(A|Phys)P(Phys) + P(A|Chem)P(Chem)
= (3/4)(1/2)+(3/5)(1/2)
= 27/40

Note that conditional probability was a means to an end in this example, not the goal itself. One reason conditional probability is important is that this is a common scenario.

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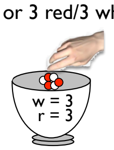
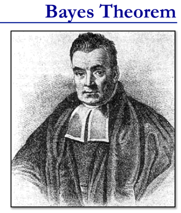
Bayes Theorem

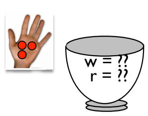
6 red or 3 red/3 white balls in an urn



Probability of drawing 3 red balls, given 3 in urn ?


Rev. Thomas Bayes c. 1701-1761

Probability of only 3 red balls in urn, given that I drew three?



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Bayes Theorem

Most common form:

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E)}$$

Expanded form (using law of total probability):

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid F^c)P(F^c)}$$

Proof:

$$P(F \mid E) = \frac{P(EF)}{P(E)} = \frac{P(E \mid F)P(F)}{P(E)}$$

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Bayes Theorem

Most common form:

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E)}$$

Expanded form (using law of total probability):

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid F^c)P(F^c)}$$

Why it's important:
Reverse conditioning
P(model | data) ~ P(data | model)
Combine new evidence (E) with prior belief (P(F))
Posterior vs prior

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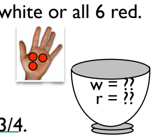
Bayes Theorem

An urn contains 6 balls, either 3 red + 3 white or all 6 red. You draw 3; all are red.
Did urn have only 3 red?



Can't tell

Suppose it was 3 + 3 with probability p=3/4.
Did urn have only 3 red?

M = urn has 3 red + 3 white
D = I drew 3 red

$P(M \mid D) = P(D \mid M)P(M)/[P(D \mid M)P(M)+ P(D \mid M^c)P(M^c)]$
$P(D \mid M) = (3 \text{ choose } 3)/(6 \text{ choose } 3) = 1/20$
$P(M \mid D) = (1/20)(3/4)/[(1/20)(3/4) + (1)(1/4)] = 3/23$

prior = 3/4 ; posterior = 3/23

I have 3 cards.

1. The first is red on both sides.
2. The second is red on one side and black on the other.
3. The third is black on both sides

I shuffle the cards and put one on the table, so you can see that the upper side is red. What is the chance that the other side is black? ½, > ½ or < ½ ?

Probability Model 1: Pick random card; put R side up if it has one.

Probability Model 2: pick random card. Pick random side to show.

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HIV testing

Suppose an HIV test is 98% effective in detecting HIV, i.e., its "false negative" rate = 2%. Suppose furthermore, the test's "false positive" rate = 1%.

0.5% of population has HIV

Let E = you test positive for HIV

Let F = you actually have HIV

What is P(F|E) ?

Solution:

$$\begin{aligned} P(F \mid E) &= \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid F^c)P(F^c)} \\ &= \frac{(0.98)(0.005)}{(0.98)(0.005) + (0.01)(1 - 0.005)} \\ &\approx 0.330 \end{aligned}$$

$P(E) \approx 1.5\%$

Note difference between conditional and joint probability: P(F|E) = 33% ; P(FE) = 0.49%

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why testing is still good

	HIV+	HIV-
Test +	0.98 = P(E|F)	0.01 = P(E|F^c)
Test -	0.02 = P(E^c|F)	0.99 = P(E^c|F^c)

Let E^c = you test **negative** for HIV

Let F = you actually have HIV

What is P(F|E^c) ?

$$\begin{aligned} P(F \mid E^c) &= \frac{P(E^c \mid F)P(F)}{P(E^c \mid F)P(F) + P(E^c \mid F^c)P(F^c)} \\ &= \frac{(0.02)(0.005)}{(0.02)(0.005) + (0.99)(1 - 0.005)} \\ &\approx 0.0001 \end{aligned}$$

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Multiple choice exam

Suppose that on a multiple choice exam, a student either knows the answer or guesses.

Let 2/3 be the probability that the student knows the answer and 1/3 the probability he guesses.

Assume that a student who guesses has probability ¼ of getting the answer right.

What is the conditional proability that a student knew the answer to a question, given that he or she answered correctly?

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summary

Conditional probability

P(E|F): Conditional probability that E occurs *given* that F has occurred.

Reduce event/sample space to points consistent w/ F (E ∩ F ; S ∩ F)

$$P(E \mid F) = \frac{P(EF)}{P(F)} \qquad (P(F) > 0)$$

$P(E \mid F) = \frac{|EF|}{|F|}$, if equiprobable outcomes.

P(EF) = P(E|F) P(F) ("the chain rule")

"P(- | F)" is a probability law, i.e., satisfies the 3 axioms

P(E) = P(E|F) P(F) + P(E|F^c) (1-P(F)) ("the law of total probability")

Bayes theorem

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E)}$$

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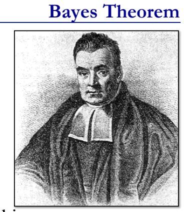
Bayes Theorem

Improbable Inspiration: The future of software may lie in the obscure theories of an 18th century cleric named Thomas Bayes



Los Angeles Times (October 28, 1996)
By Leslie Helm, Times Staff Writer

When Microsoft Senior Vice President Steve Ballmer [recent CEO] first heard his company was planning a huge investment in an Internet service offering… he went to Chairman Bill Gates with his concerns…

Gates began discussing the critical role of "Bayesian" systems…

source: http://www.ar-tiste.com/latimes_oct-96.html

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