Counting Rules, etc

- Product Rule
- Generalized Product Rule
- Division Rule
- Bijection Rule
- Sum Rule
- Combinatorial argument
- Binomial Theorem
- Pigeonhole Principle

Generalized Product Rule

- If \( S \) is a set of sequences of length \( k \) for which there are
  - \( n_1 \) choices for the first element of sequence
  - \( n_2 \) choices for the second element given any particular choice for first
  - \( n_3 \) choices for third given any particular choice for first and second.
  - …
  - Then \( |S| = n_1 \times n_2 \times \ldots \times n_k \)

When applying generalized product rule

- Have in mind a sequence of choices that produces the objects you are trying to count. (Usually there are many possibilities.)

Division Rule

- If \( f: A \rightarrow B \) is \( k \)-to-1 function, then \( |A| = k|B| \)

Example:
- \( A \) is the set of ears in the room
- \( B \) is the set of people.
- Each ear maps to exactly one person.
- Each person has exactly two ears that map to it.
- Then the number of ears is twice # people

Sum Rule

- If \( S = A \cup B \) and \( A \) and \( B \) are disjoint (mutually exclusive) then \( |S| = |A| + |B| \)

- More generally, inclusion/exclusion.
Combinatorial argument

Let $S$ be a set of objects.
- Show how to count a set in one way $\rightarrow N$
- Show how to count a set in another way $\rightarrow M$

Conclude that $N = M$

Solving Pigeonhole Principle Problems

- What are the pigeons?
- What are the pigeonholes?
- What is the rule for assigning a pigeon to a pigeonhole?

friending pigeons

There are many people in this room, some of whom are friends, some of whom are not…

Prove that some two people have the same number of friends.

counting paths

How many ways to walk from 1st and Spring to 5th and Pine only going North and East?

Instead of tracing paths on the grid above, list choices. You walk 7 blocks; at each intersection choose N or E; must choose N exactly 3 times.

\[
\binom{7}{3} = 35
\]

How many ways to walk from 1st and Spring to 5th and Pine only going North and East, if I want to stop at Starbucks on the way?

Other problems

10 people of different heights. How many ways to line up 5 of them?

Line up 5 of them in height order?
8 by 8 chessboard

- How many ways to place a pawn, bishop and knight so that none are in same row or column?

quick review of cards

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Clubs, Diamonds, Spades

counting cards

- How many possible 5 card hands? \( \binom{52}{5} \)
- A “straight” is five consecutive rank cards of any suit. How many possible straights?
  \[ 10 \cdot 4^5 = 10,240 \]
- How many flushes are there?
  \[ 4 \cdot \binom{13}{5} = 5,148 \]

more counting cards

- How many straights that are not flushes?
  \[ 10 \cdot 4^5 - 10 \cdot 4 = 10,200 \]
- How many flushes that are not straights?
  \[ 4 \cdot \binom{13}{5} - 10 \cdot 4 = 5,108 \]

the sleuth's criterion (Rudich)

For each object constructed it should be possible to reconstruct the unique sequence of choices that led to it!

Example: How many ways are there to choose a 5 card hand that contains at least 3 aces?

\[ \binom{4}{3} \cdot \binom{49}{2} \]

Choose 3 aces, then choose 2 cards from remaining 49.

When in doubt break set up into disjoint sets you know how to count!
Lessons

- Solve the same problem in different ways!
- If needed, break sets up into disjoint subsets that you know for sure how to count.
- Once you specify the sequence of choices you are making to construct the objects, make sure that given the result, you can tell exactly what choice was made at each step!

Rooks on Chessboard

- Number of ways to place 2 identical rooks on a chessboard so that they don’t share a row or column.

Doughnuts

- You go to Top Pot to buy a dozen doughnuts. Your choices today are
  - Chocolate
  - Lemon-filled
  - Sugar
  - Glazed
  - Plain
- How many ways to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

Bijection Rule

- Count one set by counting another.
- Example:
  - A: all ways to select a dozen doughnuts when five varieties are available.
  - B: all 16 bit sequences with exactly 4 ones

Other problems

- # of 7 digit numbers (decimal) with at least one repeating digit? (allowed to have leading zeros).

- # of 3 character password with at least one digit each character either digit 0-9 or letter a-z.

  10 36 36 + 36 10 36 + 36 36 10

Bijection between A and B

- A: all ways to select a dozen doughnuts when five varieties are available.
- B: all 16 bit sequences with exactly 4 ones

0011000000100100
Mapping from doughnuts to bit strings

e chocolate, l lemon-filled, s sugar, g glazed, and p plain

to the sequence:

\[
\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Buying 2 dozen bagels

• Choosing from 3 varieties:
  – Plain
  – Garlic
  – Pumpernickel

• How many ways to grab 2 dozen if you want at least 3 of each type and bagels of the same type are indistinguishable.

Solve the following problems using the pigeonhole principle. For each problem, try to identify the pigeons, the pigeonholes, and a rule assigning each pigeon to a pigeonhole.

(a) In a certain Institute of Technology, Every ID number starts with a 9. Suppose that each of the 75 students in a class sums the nine digits of their ID number. Explain why two people must arrive at the same sum.

(b) In every set of 100 integers, there exist two whose difference is a multiple of 37.