Linearity, II

Let X and Y be two random variables derived from outcomes of a *single* experiment. Then

$$\boxed{E[X+Y] = E[X] + E[Y]}$$

Can extend by induction to say that

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

expectation of sum = sum of expectations

Given a DNA sequence of length n

e.g. AAATGAATGAATCC……

where each position is

A with probability p_A

T with probability p_T

G with probability p_G

C with probability p_C.

What is the expected number of occurrences of the substring AATGAAT?

AAATGAATGAATCC

AAATGAATGAATCC

variance

Definitions

The *variance* of a random variable X with mean $E[X] = \mu$ is

$$\mathrm{Var}[X] = E[(X-\mu)^2],$$

often denoted σ^2.

The *standard deviation* of X is

$$\sigma = \sqrt{\mathrm{Var}[X]}$$

$$\boxed{\mathrm{Var}[aX+b] = a^2\,\mathrm{Var}[X]}$$

NOT linear;
insensitive to location (b),
quadratic in scale (a)

$$\boxed{\mathrm{Var}[aX+b] = a^2\,\mathrm{Var}[X]}$$

NOT linear;
insensitive to location (b),
quadratic in scale (a)

$$\begin{aligned}\mathrm{Var}(aX + b) &= E[(aX + b - a\mu - b)^2]\\ &= E[a^2(X - \mu)^2]\\ &= a^2E[(X - \mu)^2]\\ &= a^2\mathrm{Var}(X)\end{aligned}$$

$$\boxed{\mathrm{Var}[aX+b] = a^2\,\mathrm{Var}[X]}$$

NOT linear;
insensitive to location (b),
quadratic in scale (a)

$$\begin{aligned}\mathrm{Var}(aX + b) &= E[(aX + b - a\mu - b)^2]\\ &= E[a^2(X - \mu)^2]\\ &= a^2E[(X - \mu)^2]\\ &= a^2\mathrm{Var}(X)\end{aligned}$$

Ex:

$$X = \begin{cases} +1 & \text{if Heads} \\ -1 & \text{if Tails} \end{cases}$$

E[X] = 0
Var[X] = 1

$$Y = \begin{cases} +1000 & \text{if Heads} \\ -1000 & \text{if Tails} \end{cases}$$

Y = 1000 X

E[Y] = E[1000 X] = 1000 E[X] = 0

Var[Y] = Var[10^3 X]=10^6Var[X] = 10^6

$$\boxed{\mathrm{Var}(X) = E[X^2] - (E[X])^2}$$

$$\boxed{\mathrm{Var}(X) = E[X^2] - (E[X])^2}$$

$$\begin{aligned} \mathrm{Var}(X) &= E[(X-\mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - \mu^2 \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

properties of variance

Example: What is Var[X] when X is outcome of one fair die?

$$E[X^2] = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{6}\right)$$
$$= \left(\frac{1}{6}\right)(91)$$

E[X] = 7/2, so $\mathrm{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$

properties of variance

In general:

$\boxed{\mathrm{Var}[X+Y] \neq \mathrm{Var}[X] + \mathrm{Var}[Y]}$ *NOT* linear

Ex 1:

Let X = ±1 based on 1 coin flip; Y=-X

properties of variance

In general:

$\boxed{\mathrm{Var}[X+Y] \neq \mathrm{Var}[X] + \mathrm{Var}[Y]}$ *NOT* linear

Ex 1:

Let X = ±1 based on 1 coin flip; Y=-X

Ex 2:

As another example, is Var[X+X] = 2Var[X]?

properties of variance

In general:

$$\boxed{\mathrm{Var}[X+Y] \neq \mathrm{Var}[X] + \mathrm{Var}[Y]}$$

NOT linear

Ex 1:

Let X = ±1 based on 1 coin flip

As shown above, E[X] = 0, Var[X] = 1

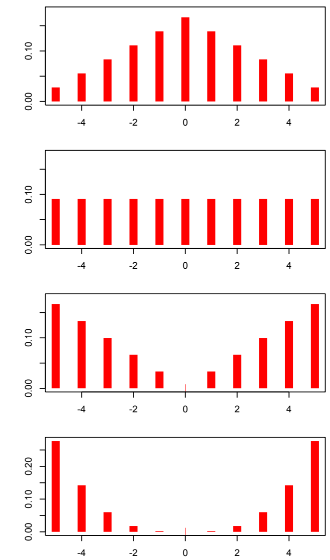
Let Y = -X; then Var[Y] = $(-1)^2$Var[X] = 1

But X+Y = 0, always, so Var[X+Y] = 0

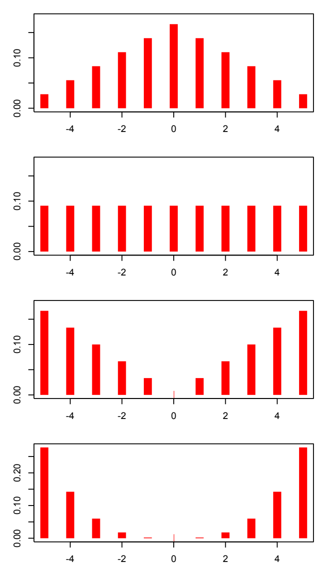
Ex 2:

As another example, is Var[X+X] = 2Var[X]?

more variance examples



more variance examples

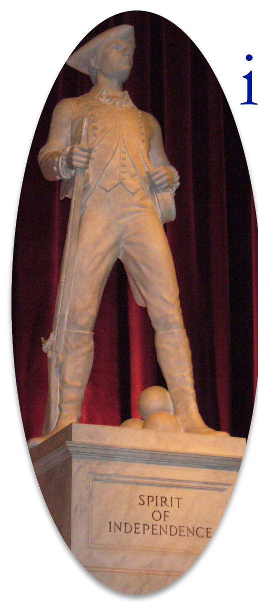


σ^2 = 5.83

σ^2 = 10

σ^2 = 15

σ^2 = 19.7



independence

# of r.v.s

Defn: r.v. X and event E are independent if the event E is independent of the event {X=x} (for any fixed x), i.e.

$\forall x\ P(\{X = x\}\ \&\ E) = P(\{X=x\}) \cdot P(E)$

Defn: r.v. X and event E are independent if the event E is independent of the event {X=x} (for any fixed x), i.e.

$\forall x\ P(\{X = x\}\ \&\ E) = P(\{X=x\}) \cdot P(E)$

Defn: Two r.v.s X and Y are independent if the events {X=x} and {Y=y} are independent (for any fixed x, y), i.e.

$\forall x, y\ P(\{X = x\}\ \&\ \{Y=y\}) = P(\{X=x\}) \cdot P(\{Y=y\})$

Defn: r.v. X and event E are independent if the event E is independent of the event {X=x} (for any fixed x), i.e.

$\forall x\ P(\{X = x\}\ \&\ E) = P(\{X=x\}) \cdot P(E)$

Defn: Two r.v.s X and Y are independent if the events {X=x} and {Y=y} are independent (for any fixed x, y), i.e.

$\forall x, y\ P(\{X = x\}\ \&\ \{Y=y\}) = P(\{X=x\}) \cdot P(\{Y=y\})$

Intuition as before: knowing X doesn't help you guess Y or E and vice versa.

Two random variables X and Y are independent if the events {X=x} and {Y=y} are independent (for any x, y), i.e.

$\forall x, y\ P(\{X = x\}\ \&\ \{Y=y\}) = P(\{X=x\}) \cdot P(\{Y=y\})$

Ex: Let X be number of heads in first n of 2n coin flips, Y be number in the last n flips, and let Z be the total. X and Y are independent:

Two random variables X and Y are independent if the events {X=x} and {Y=y} are independent (for any x, y), i.e.

$\forall$x, y P({X = x} & {Y=y}) = P({X=x}) • P({Y=y})

Ex: Let X be number of heads in first n of 2n coin flips, Y be number in the last n flips, and let Z be the total. X and Y are independent:

$$P(X=j) = \binom{n}{j} 2^{-n}$$
$$P(Y=k) = \binom{n}{k} 2^{-n}$$
$$P(X=j \wedge Y=k) = \binom{n}{j}\binom{n}{k} 2^{-2n} = P(X=j)P(Y=k)$$

But X and Z are *not* independent, since, e.g., knowing that X = 0 precludes Z > n. E.g., P(X = 0) and P(Z = n+1) are both positive, but P(X = 0 & Z = n+1) = 0.

Defn: Two r.v.s X and Y are independent if the events {X=x} and {Y=y} are independent (for any fixed x, y), i.e.

$\forall$x, y P({X = x} & {Y=y}) = P({X=x}) • P({Y=y})

Theorem: If X & Y are *independent*, then E[X•Y] = E[X]•E[Y]

Note: *NOT* true in general; see earlier example $E[X^2] \neq E[X]^2$

Theorem: If X & Y are *independent*, then E[X•Y] = E[X]•E[Y]

Proof:

Let $x_i, y_i, i = 1, 2, \ldots$ be the possible values of X, Y.

$$\begin{aligned} E[X \cdot Y] &= \sum_i \sum_j x_i \cdot y_j \cdot P(X = x_i \wedge Y = y_j) \\ &= \sum_i \sum_j x_i \cdot y_j \cdot P(X = x_i) \cdot P(Y = y_j) \\ &= \sum_i x_i \cdot P(X = x_i) \cdot \left(\sum_j y_j \cdot P(Y = y_j) \right) \\ &= E[X] \cdot E[Y] \end{aligned}$$

independence

Note: *NOT* true in general; see earlier example $E[X^2] \neq E[X]^2$

properties of variance

In general:

$$\boxed{Var[X+Y] \neq Var[X] + Var[Y]}$$

NOT linear

variance of independent r.v.s is additive

(Bienaymé, 1853)

Theorem: If X & Y are *independent*, then

$$Var[X+Y] = Var[X]+Var[Y]$$

variance of independent r.v.s is additive

(Bienaymé, 1853)

Theorem: If X & Y are *independent*, then

$$Var[X+Y] = Var[X]+Var[Y]$$

Proof:

$$Var[X+Y]$$

$$= E[(X+Y)^2] - (E[X+Y])^2$$

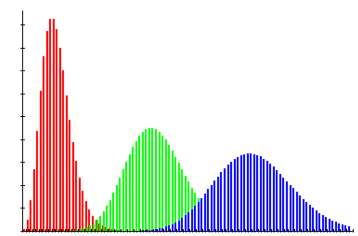
$$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$$

$$= E[X^2] + 2E[XY] + E[Y^2] - ((E[X])^2 + 2E[X]E[Y] + (E[Y])^2)$$

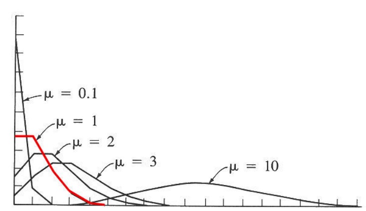
$$= E[X^2] - (E[X])^2 + E[Y^2] - (E[Y])^2 + 2(E[XY] - E[X]E[Y])$$

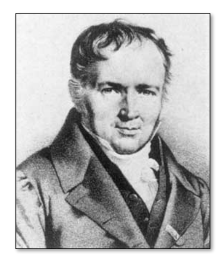
$$= Var[X] + Var[Y] + 2(E[X]E[Y] - E[X]E[Y])$$

$$= Var[X] + Var[Y]$$



a zoo of (discrete) random variables





discrete uniform random variables

A discrete random variable X equally likely to take any (integer) value between integers a and b, inclusive, is *uniform*.

Notation: $X \sim \text{Unif}(a,b)$

Probability: $P(X = i) = \dfrac{1}{b - a + 1}$

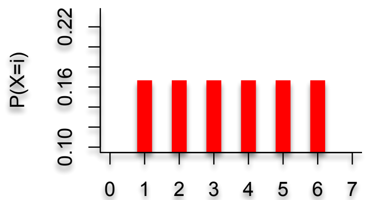
Mean, Variance: $E[X] = \dfrac{a+b}{2}, \text{Var}[X] = \dfrac{(b-a)(b-a+2)}{12}$

Example: value shown on one roll of a fair die is Unif(1,6):

$P(X=i) = 1/6$

$E[X] = 7/2$

$\text{Var}[X] = 35/12$



Bernoulli random variables

An experiment results in "Success" or "Failure"

X is an *indicator random variable* (1 = success, 0 = failure)

P(X=1) = p and P(X=0) = 1-p

X is called a *Bernoulli* random variable: X ~ Ber(p)

E[X] =

Var(X) = $E[X^2] - (E[X])^2$ =

Bernoulli random variables

An experiment results in "Success" or "Failure"

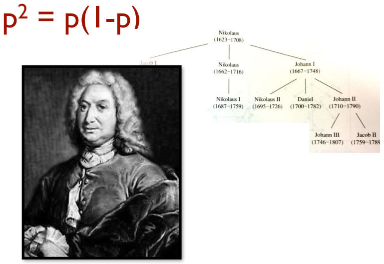
X is an *indicator random variable* (1 = success, 0 = failure)

P(X=1) = p and P(X=0) = 1-p

X is called a *Bernoulli* random variable: X ~ Ber(p)

$E[X] = E[X^2] = p$

$\text{Var}(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$



Jacob (aka James, Jacques) Bernoulli, 1654 – 1705

Examples:

- coin flip
- random binary digit
- whether a disk drive crashed

binomial random variables

Consider n independent random variables $Y_i \sim \text{Ber}(p)$

$X = \Sigma_i Y_i$ is the number of successes in n trials

X is a *Binomial* random variable: X ~ Bin(n,p)

Pr (X=k) = ?

E(X) = ?

Var(X) = ?

binomial random variables

Consider n independent random variables $Y_i \sim$ Ber(p)

$X = \Sigma_i Y_i$ is the number of successes in n trials

X is a *Binomial* random variable: X ~ Bin(n,p)

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i} \quad i = 0, 1, \ldots, n$$

By Binomial theorem, $\sum_{i=0}^{n} P(X = i) = 1$

E[X] = pn

Var(X) = p(1-p)n

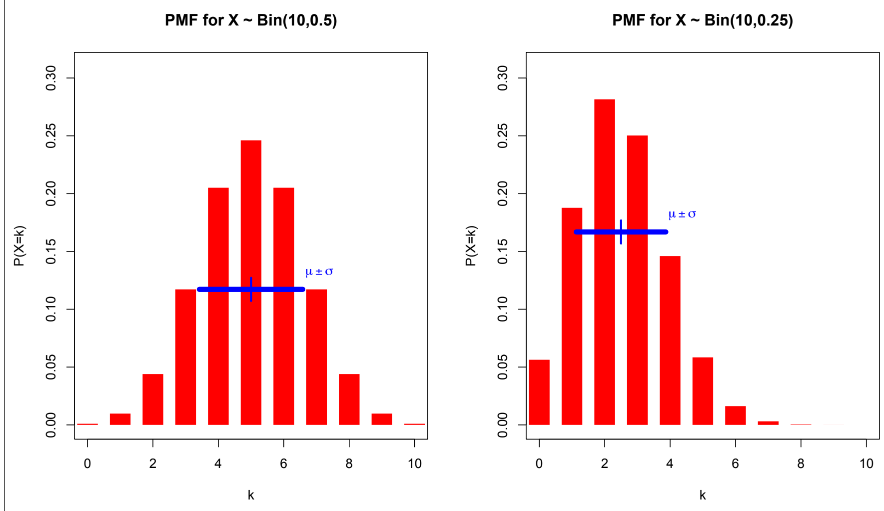
Examples

# of heads in n coin flips

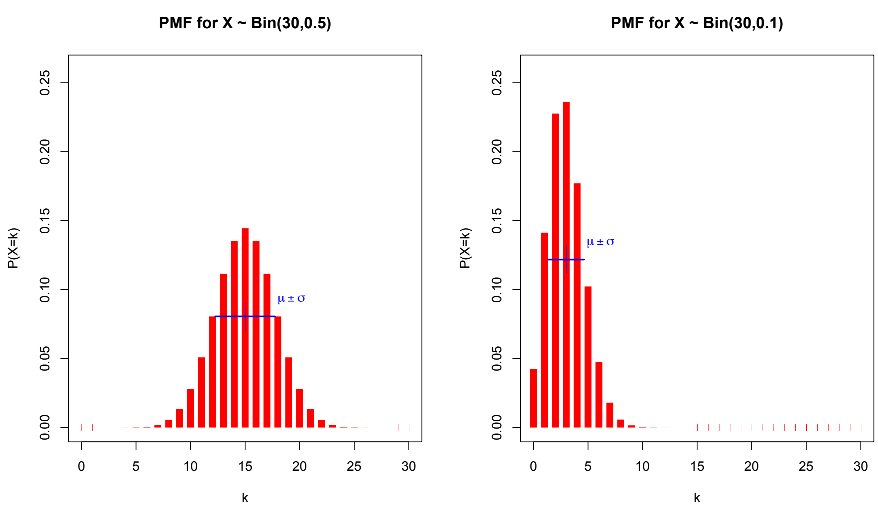
# of 1's in a randomly generated length n bit string

# of disk drive crashes in a 1000 computer cluster

binomial pmfs



binomial pmfs



mean, variance of the binomial (II)

If $Y_1, Y_2, \ldots, Y_n \sim$ Ber(p) and independent,

then $X = \sum_{i=1}^{n} Y_i \sim$ Bin(n, p).

$\boxed{E[X] = np}$

$$E[X] = E\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} E[Y_i] = nE[Y_1] = np$$

$\boxed{\text{Var}[X] = np(1-p)}$

$$\text{Var}[X] = \text{Var}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \text{Var}[Y_i] = n\text{Var}[Y_1] = np(1-p)$$