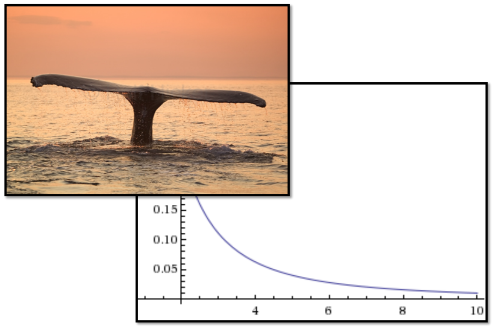
tail bounds



Markov's inequality

Suppose we know that X is always non-negative.

Theorem: If X is a non-negative random variable, then for every $\alpha > 0$, we have

$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

Corr:

$$P(X \geq cE[X]) \leq 1/c$$

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Markov's inequality

Theorem: If X is a non-negative random variable, then for every $\alpha > 0$, we have

$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

Proof:

$$\begin{aligned} E[X] &= \Sigma_x \, x \, p(x) \\ &= \Sigma_{x<\alpha} \, x \, p(x) + \Sigma_{x \geq \alpha} \, x \, p(x) \\ &\geq \quad 0 \quad + \Sigma_{x \geq \alpha} \, \alpha \, p(x) \qquad (x \geq 0;\ \alpha \leq x) \\ &= \alpha P(X \geq \alpha) \end{aligned}$$

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Markov's inequality

Theorem: If X is a non-negative random variable, then for every $\alpha > 0$, we have

$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

Example: if X = daily advertising expenses and E[X] = 1500

Then, by Markov's inequality,

$$P(X \geq 6000) \leq \frac{1500}{6000} = 0.25$$

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Chebyshev's inequality

If we know *more* about a random variable, we can often use that to get *better* tail bounds.

Suppose we *also* know the variance.

Theorem: If Y is an arbitrary random variable with $E[Y] = \mu$, then, for any $\alpha > 0$,

$$P(|Y - \mu| \geq \alpha) \leq \frac{\text{Var}[Y]}{\alpha^2}$$

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Chebyshev's inequality

Theorem: If Y is an arbitrary random variable with $\mu = E[Y]$, then, for any $\alpha > 0$,

$$P(|Y - \mu| \geq \alpha) \leq \frac{\text{Var}[Y]}{\alpha^2}$$

Proof: Let $X = (Y - \mu)^2$

X is non-negative, so we can apply Markov's inequality:

$$P(|Y - \mu| \geq \alpha) = P(X \geq \alpha^2)$$
$$\leq \frac{E[X]}{\alpha^2} = \frac{\text{Var}[Y]}{\alpha^2}$$

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Chebyshev's inequality

$$P(|Y-\mu| \geq \alpha) \leq \frac{\mathrm{Var}[Y]}{\alpha^2}$$

E.g., suppose:

Y = money spent on advertising in a day
E[Y] = 1500
Var[Y] = 500^2 (i.e. SD[Y] = 500)

$$P(Y \geq 6000) = P(|Y-\mu| \geq 4500)$$

$$\leq \frac{500^2}{4500^2} = \frac{1}{81} \approx 0.012$$

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Chebyshev's inequality

Theorem: If Y is an arbitrary random variable with μ = E[Y], then, for any $\alpha > 0$,

$$P(|Y-\mu| \geq \alpha) \leq \frac{\mathrm{Var}[Y]}{\alpha^2}$$

Corr: If

$$\sigma = SD[Y] = \sqrt{\mathrm{Var}[Y]}$$

Then:

$$P(|Y-\mu| \geq t\sigma) \leq \frac{\sigma^2}{t^2\sigma^2} = \frac{1}{t^2}$$

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super strong tail bounds

Y ~ Bin(15000, 0.1)
μ = E[Y] = 1500, $\sigma = \sqrt{\mathrm{Var(Y)}}$ = 36.7

P(Y ≥ 6000) = P(Y ≥ 4 μ) ≤ ¼ (Markov)
P(Y ≥ 6000) = P(Y- μ ≥ 122 σ) ≤ 7x10^{-5} (Chebyshev)

Poisson approximation: Y ~ Poi(1500)
Rough computer calculation:

P(Y ≥ 6000) ≪ 10^{-1600}

And the exact value is ≈ 4 x 10^{-2031}

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