From 15-251

Great Theoretical Ideas in Computer Science

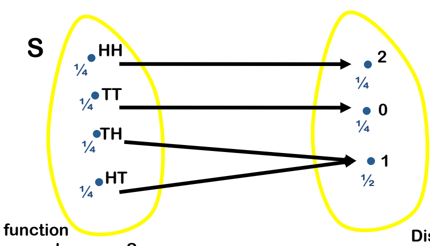
CMU (with modifications)

Random variables

Discrete random variable: take a finite or countable number of values

Two Coins Tossed

X: {TT, TH, HT, HH} → {0, 1, 2} counts the number of heads



X is a function on the sample space S That takes a random input

Distribution on the reals!

Two Coins Tossed

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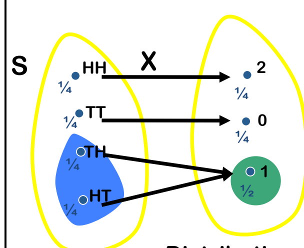


Distribution on X

$Pr(X = a) = Pr(\{t \in S \mid X(t) = a\})$

$Pr(X = 1)$

$= Pr(\{t \in S \mid X(t) = 1\})$

$= Pr(\{TH, HT\}) = ½$

Definition: Expectation

The expectation, or expected value of a random variable X is written as E[X], and is

$$E[X] = \sum_{t \in S} Pr(t)\, X(t) = \sum_{k} k\, Pr[X = k]$$

X is a function on the sample space S

X has a prob. distribution on its values

(assuming X takes values in the naturals)

A Quick Calculation…

What if I flip a coin 3 times? What is the expected number of heads?

$E[X] = (1/8)\times0 + (3/8)\times1 + (3/8)\times2 + (1/8)\times3 = 1.5$

But $Pr[X = 1.5] = 0$

Moral: don't always expect the expected. $Pr[X = E[X]]$ may be 0 !

Functions of a R.V.

Suppose that X is a random variable defined on probability space (S, p(.))
Then Y= g(X) is also a random variable on the same probability space (if defined everywhere).

E.g., $Y = X^2$

$Z = 3^X + 2$

Expectation of a function of a r.v.

$$E(X) = \sum_k kPr(X = k) \; = \sum_{\omega \in S} X(\omega)Pr(\omega)$$

$$E(g(X)) = \sum_k g(k)Pr(X = k)$$

k in range of X

Operations on R.V.s

You can define any random variable you want on a probability space.
Given a collection of random variables, you can sum them, take differences, or do most other math operations…

E.g., (X + Y)(t) = X(t) + Y(t)

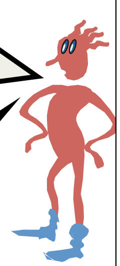
(X*Y)(t) = X(t) * Y(t)

$(X^Y)(t) = X(t)^{Y(t)}$

Random variables and expectations allow us to give elegant solutions to problems that seem really really messy…

If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?

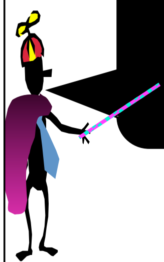
On average, in class of size m, how many pairs of people will have the same birthday?



Pretty messy with direct counting…

The new tool is called "Linearity of Expectation"

Linearity of Expectation



If Z = X+Y, then

$E[Z] = E[X] + E[Y]$

Without any conditions on X and Y

By Induction

$E[X_1 + X_2 + \ldots + X_n] =$

$E[X_1] + E[X_2] + \ldots. + E[X_n]$

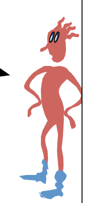
The expectation of the sum

=

The sum of the expectations

Let's see why
Linearity of Expectation
is so useful...

If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?



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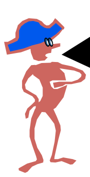
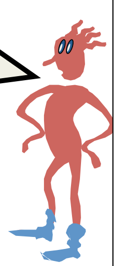




Hmm…

$\sum_k$ k Pr(exactly k letters end up in correct envelopes)

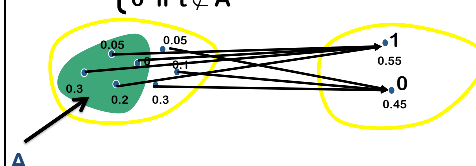
= $\sum_k$ k (...aargh!!...)

Indicator Random Variables

For any event A, can define the "indicator random variable" for event A:

$$X_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases}$$

$\boxed{E[X_A] = 1 \times Pr(X_A = 1) = Pr(A)}$



Use Linearity of Expectation

Let A_i be the event the i^{th} letter ends up in its correct envelope

Let X_i be the "indicator" R.V. for A_i

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = X_1 + \ldots + X_{100}$

We are asking for E[Z]

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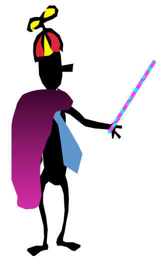
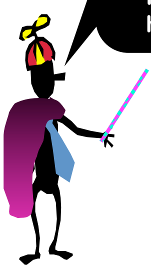


Let $Z = X_1 + \ldots + X_{100}$

We are asking for E[Z]

$E[X_i] = Pr(A_i) = 1/100$

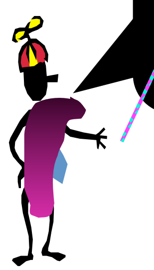
So $E[Z] = 1$



So, in expectation, 1 letter will be in the same correct envelope

Pretty neat: it doesn't depend on how many letters!

Use Linearity of Expectation



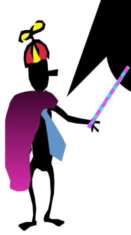
General approach:

View thing you care about as expected value of some RV

Write this RV as sum of simpler RVs

Solve for their expectations and add them up!

Ex. #2



We flip n coins of bias p. What is the expected number of heads?

We could do this by summing

$$\sum_k k Pr(X = k) = \sum_k k \binom{n}{k} p^k (1-p)^{n-k}$$

But now we know a better way!

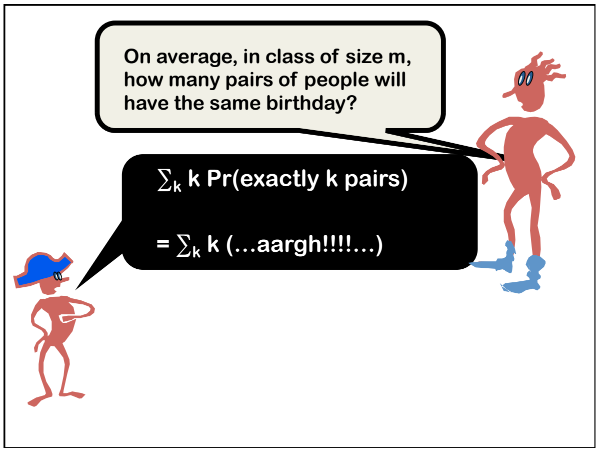
Linearity of Expectation!

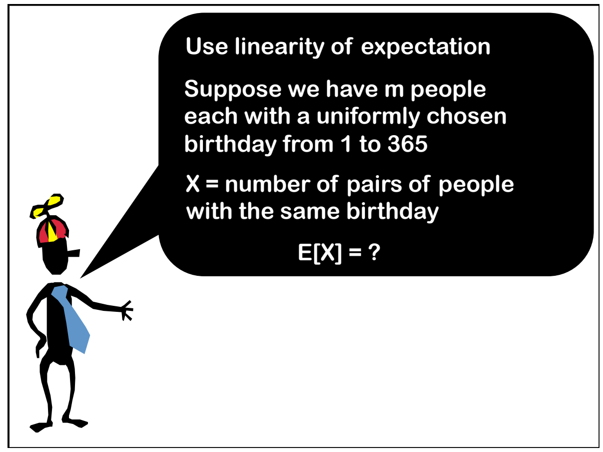
Let X = number of heads when n independent coins of bias p are flipped

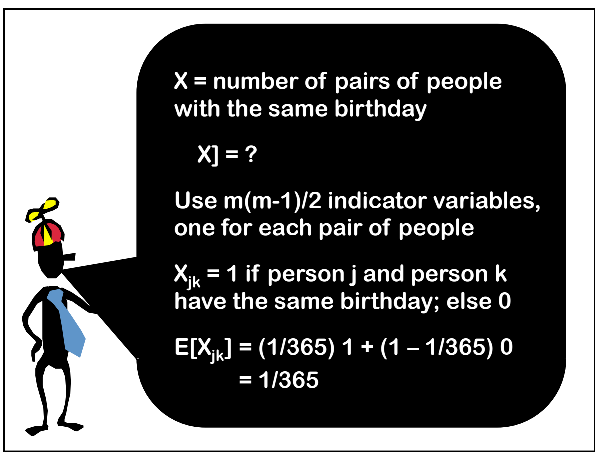
Break X into n simpler RVs:

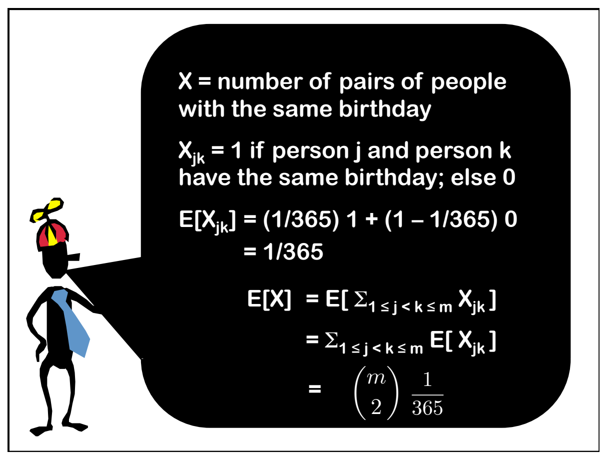
$$X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ coin is heads} \\ 0 & \text{if the } i^{th} \text{ coin is tails} \end{cases}$$

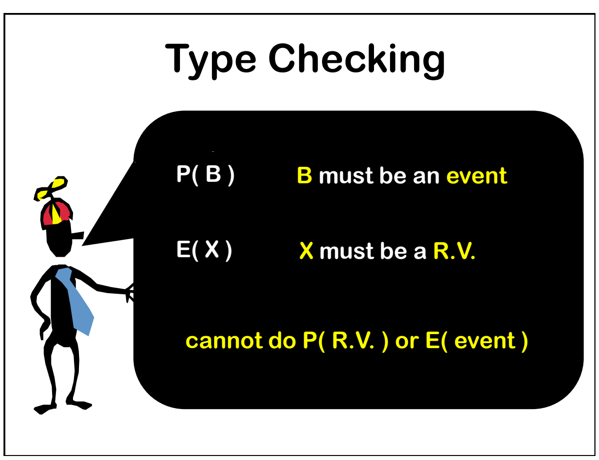
$E[X] = E[\Sigma_i X_i] = \Sigma_i E[X_i] = \Sigma_i p = np$


On average, in class of size m, how many pairs of people will have the same birthday?
$\sum_k$ k Pr(exactly k pairs)
= $\sum_k$ k (...aargh!!!!...)


Use linearity of expectation
Suppose we have m people each with a uniformly chosen birthday from 1 to 365
X = number of pairs of people with the same birthday
E[X] = ?


X = number of pairs of people with the same birthday
X] = ?
Use m(m-1)/2 indicator variables, one for each pair of people
X_{jk} = 1 if person j and person k have the same birthday; else 0
$E[X_{jk}] = (1/365)\ 1 + (1 - 1/365)\ 0$
$= 1/365$


X = number of pairs of people with the same birthday
X_{jk} = 1 if person j and person k have the same birthday; else 0
$E[X_{jk}] = (1/365)\ 1 + (1 - 1/365)\ 0$
$= 1/365$
$E[X] = E[\ \Sigma_{1 \le j < k \le m} X_{jk}\ ]$
$= \Sigma_{1 \le j < k \le m} E[\ X_{jk}\ ]$
$= \binom{m}{2} \frac{1}{365}$


Type Checking
P(B)
B must be an event
E(X)
X must be a R.V.
cannot do P(R.V.) or E(event)