joint distributions

Often, several random variables are *simultaneously* observed

X = height and Y = weight

X = cholesterol and Y = blood pressure

X_1, X_2, X_3 = work loads on servers A, B, C

Joint probability mass function:

$f_{XY}(x, y) = P(\{X = x\} \& \{Y = y\})$

Joint cumulative distribution function:

$F_{XY}(x, y) = P(\{X \le x\} \& \{Y \le y\})$

examples

Two joint PMFs

W \ Z	1	2	3
1	2/24	2/24	2/24
2	2/24	2/24	2/24
3	2/24	2/24	2/24
4	2/24	2/24	2/24

X \ Y	1	2	3
1	4/24	1/24	1/24
2	0	3/24	3/24
3	0	4/24	2/24
4	4/24	0	2/24

$P(W = Z) = 3 * 2/24 = 6/24$

$P(X = Y) = (4 + 3 + 2)/24 = 9/24$

Can look at arbitrary relationships among variables this way

marginal distributions

Two joint PMFs

W \ Z	1	2	3	$f_W(w)$
1	2/24	2/24	2/24	6/24
2	2/24	2/24	2/24	6/24
3	2/24	2/24	2/24	6/24
4	2/24	2/24	2/24	6/24
$f_Z(z)$	8/24	8/24	8/24	

X \ Y	1	2	3	$f_X(x)$
1	4/24	1/24	1/24	6/24
2	0	3/24	3/24	6/24
3	0	4/24	2/24	6/24
4	4/24	0	2/24	6/24
$f_Y(y)$	8/24	8/24	8/24	

Marginal PMF of one r.v.: sum over the other (Law of total probability)

$f_Y(y) = \Sigma_x f_{XY}(x,y)$

$f_X(x) = \Sigma_y f_{XY}(x,y)$

Question: Are W & Z independent? Are X & Y independent?

joint, marginals and independence

Repeating the Definition: Two random variables X and Y are independent if the events {X=x} and {Y=y} are independent (for any fixed x, y), i.e.

$\forall x, y \; P(\{X = x\} \& \{Y=y\}) = P(\{X=x\}) \cdot P(\{Y=y\})$

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$\forall$x, y P({X = x} & {Y=y}) = P({X=x}) • P({Y=y})

Equivalent Definition: Two random variables X and Y are independent if their *joint* probability mass function is the product of their *marginal* distributions, i.e.

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Exercise: Show that this is also true of their *cumulative* distribution functions

expectation of a function of 2 r.v.'s

A function g(X,Y) defines a new random variable.

Its expectation is:

$E[g(X, Y)] = \Sigma_x \Sigma_y \, g(x, y) \, f_{XY}(x,y)$

Expectation is linear. E.g., if g is linear:

$E[g(X, Y)] = E[a\,X + b\,Y + c] = a\,E[X] + b\,E[Y] + c$

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Example:

g(X,Y) = 2X-Y

E[g(X,Y)] = 72/24 = 3

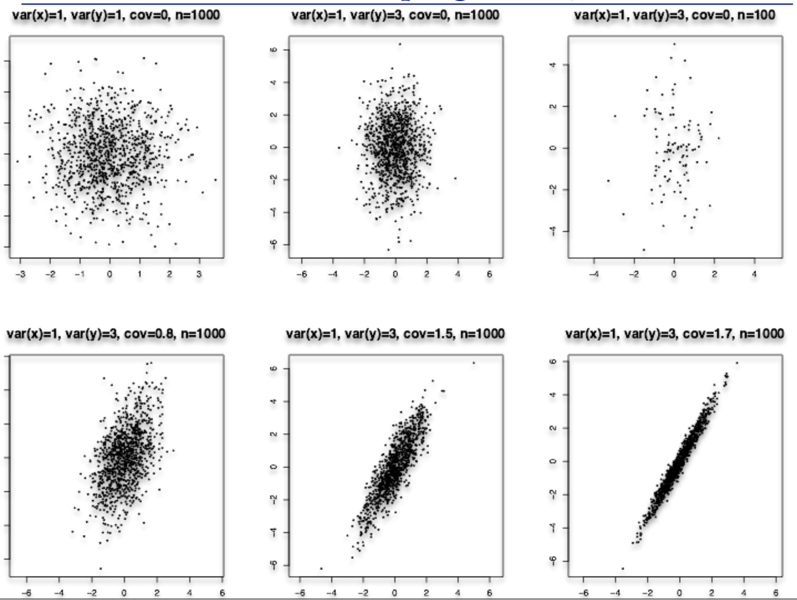
E[g(X,Y)] = 2•E[X] - E[Y]

= 2•2.5 - 2 = 3

X \ Y	1	2	3
1	1 • 4/24	0 • 1/24	-1 • 1/24
2	3 • 0/24	2 • 3/24	1 • 3/24
3	5 • 0/24	4 • 4/24	3 • 2/24
4	7 • 4/24	6 • 0/24	5 • 2/24

recall both marginals are uniform

sampling from a joint distribution



random variables – summary

RV: a numeric function of the outcome of an experiment

Probability Mass Function $p(x)$: prob that RV $= x$; $\Sigma p(x)=1$

Cumulative Distribution Function $F(x)$: *probability that RV* $\leq x$

Generalize to *joint* distributions; *independence* & *marginals*

Expectation:

mean, average, "center of mass," fair price for a game of chance

of a random variable: $E[X] = \Sigma_x\, xp(x)$

of a function: if $Y = g(X)$, then $E[Y] = \Sigma_x\, g(x)p(x)$

(probability)-weighted average

linearity:

$E[aX + b] = aE[X] + b$

$E[X+Y] = E[X] + E[Y]$; even if dependent

this interchange of "order of operations" is quite special to linear combinations. E.g., $E[XY]\neq E[X]\bullet E[Y]$, *in general (but see below)*

random variables – summary

Conditional Expectation:

$E[X \mid A] = \sum_x x\bullet P(X=x \mid A)$

Law of Total Expectation

$E[X] = E[X \mid A]\bullet P(A) + E[X \mid \neg A]\bullet P(\neg A)$

Variance:

$Var[X] = E[\,(X-E[X])^2\,] = E[X^2] - (E[X])^2$

Standard deviation: $\sigma = \sqrt{\mathrm{Var}[X]}$

$\mathrm{Var}[aX+b] = a^2\,\mathrm{Var}[X]$ "Variance is insensitive to location, quadratic in scale"

If X & Y are *independent*, then

$E[X\bullet Y] = E[X]\bullet E[Y]$

$\mathrm{Var}[X+Y] = \mathrm{Var}[X]+\mathrm{Var}[Y]$

(These two equalities hold for *indp* rv's; but not in general.)

random variables – summary

Important Examples:

Uniform(a,b): $P(X=i) = \dfrac{1}{b-a+1}$ $\quad \mu = \dfrac{a+b}{2}, \sigma^2 = \dfrac{(b-a)(b-a+2)}{12}$

Bernoulli: $P(X = 1) = p$, $P(X = 0) = 1\text{-}p$ $\quad \mu = p,\ \ \sigma^2 = p(1\text{-}p)$

Binomial: $P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$ $\quad \mu = np,\ \sigma^2 = np(1\text{-}p)$

Poisson: $P(X=i) = e^{-\lambda}\frac{\lambda^i}{i!}$ $\quad \mu = \lambda,\ \ \sigma^2 = \lambda$

$Bin(n,p) \approx Poi(\lambda)$ where $\lambda = np$ fixed, $n \to \infty$ (and so $p=\lambda/n \to 0$)

Geometric $P(X = k) = (1\text{-}p)^{k\text{-}1}p$ $\quad \mu = 1/p,\ \sigma^2 = (1\text{-}p)/p^2$

Many others, e.g., hypergeometric, negative binomial, …