more on expectation

Linearity of expectation, I

For any constants *a*, *b*: $\boxed{E[aX + b] = aE[X] + b}$

Linearity, II

Let X and Y be two random variables derived from outcomes of a *single* experiment. Then

$$\boxed{E[X+Y] = E[X] + E[Y]}$$

Can extend by induction to say that

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

expectation of sum = sum of expectations

Note: Linearity is special!

It is *not* true in general that

$E[X \cdot Y] = E[X] \cdot E[Y]$

$E[X^2] = E[X]^2$

$E[X/Y] = E[X] / E[Y]$

$E[\operatorname{asinh}(X)] = \operatorname{asinh}(E[X])$

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variance

Alice & Bob are gambling. X = Alice's gain per flip:

$$X = \begin{cases} +1 & \text{if Heads} \\ -1 & \text{if Tails} \end{cases}$$

E[X] = 0

. . . Time passes . . .

Alice (yawning) says "let's raise the stakes"

$$Y = \begin{cases} +1000 & \text{if Heads} \\ -1000 & \text{if Tails} \end{cases}$$

E[Y] = 0, as before.

Are you (Bob) equally happy to play the new game?

E[X] measures the "average" or "central tendency" of X.

What about its *variability*?

Definitions

The *variance* of a random variable X with mean E[X] = μ is

$$\mathrm{Var}[X] = E[(X-\mu)^2],$$

often denoted σ^2.

The *standard deviation* of X is

$$\sigma = \sqrt{\mathrm{Var}[X]}$$

Alice & Bob are gambling (again). X = Alice's gain per flip:

$$X = \begin{cases} +1 & \text{if Heads} \\ -1 & \text{if Tails} \end{cases}$$

E[X] = 0 $\quad\quad$ Var[X] = 1

... Time passes ...

Alice (yawning) says "let's raise the stakes"

$$Y = \begin{cases} +1000 & \text{if Heads} \\ -1000 & \text{if Tails} \end{cases}$$

E[Y] = 0, as before. $\quad\quad$ Var[Y] = 1,000,000

Are you (Bob) equally happy to play the new game?

The *variance* of a r.v. X with mean E[X] = μ is

Var[X] = E[(X-μ)^2],

often denoted σ^2.

$$E(X) = \sum_{k \in Range(X)} k Pr(X = k)$$

$$E(g(X)) = \sum_{j \in Range(g(X))} j Pr(g(X) = j)$$

$$= \sum_{k \in Range(X)} g(k) Pr(X = k)$$

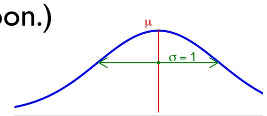
Prob	outcome	X
1/6	123	3
1/6	132	1
1/6	213	1
1/6	231	0
1/6	312	0
1/6	321	1

The *variance* of a random variable X with mean E[X] = μ is Var[X] = E[(X-μ)^2], often denoted σ^2.

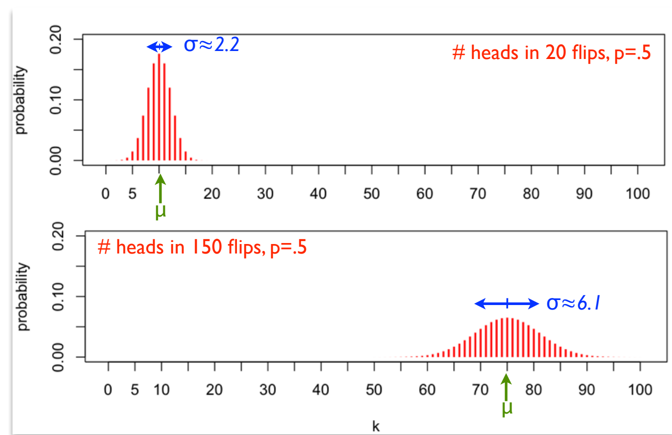
I: Square always ≥ 0, and exaggerated as X moves away from μ, so Var[X] emphasizes *deviation* from the mean.

II: Numbers vary a lot depending on exact distribution of X, but it is common that X is
within μ ± σ ~66% of the time, and
within μ ± 2σ ~95% of the time.

(We'll see the reasons for this soon.)



μ = E[X] is about *location*; σ = $\sqrt{\text{Var}(X)}$ is about *spread*



Blue arrows denote the interval μ ± σ

(and note σ bigger in absolute terms in second ex., but smaller as a proportion of μ or max.)

Two games:

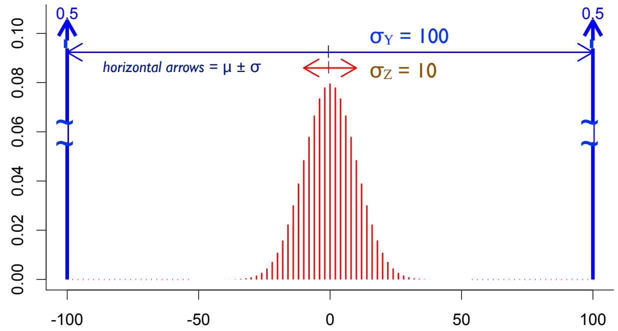
a) flip 1 coin, win Y = \$100 if heads, \$-100 if tails

b) flip 100 coins, win Z = (#(heads) - #(tails)) dollars

Same expectation in both: E[Y] = E[Z] = 0

Same extremes in both: max gain = \$100; max loss = \$100

But *variability* is very different:



$\boxed{\text{Var}[aX+b] = a^2\,\text{Var}[X]}$

NOT linear;
insensitive to location (b),
quadratic in scale (a)

$$\begin{aligned}\text{Var}(aX + b) &= E[(aX + b - a\mu - b)^2] \\ &= E[a^2(X - \mu)^2] \\ &= a^2E[(X - \mu)^2] \\ &= a^2\text{Var}(X)\end{aligned}$$

Ex:

$$X = \begin{cases} +1 & \text{if Heads} \\ -1 & \text{if Tails} \end{cases}$$

E[X] = 0
Var[X] = 1

$$Y = \begin{cases} +1000 & \text{if Heads} \\ -1000 & \text{if Tails} \end{cases}$$

Y = 1000 X

E[Y] = E[1000 X] = 1000 E[X] = 0

Var[Y] = Var[10^3 X]=10^6Var[X] = 10^6

$$\boxed{\text{Var}(X) = E[X^2] - (E[X])^2}$$

$$\begin{aligned}\text{Var}(X) &= E[(X-\mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - \mu^2 \\ &= E[X^2] - (E[X])^2\end{aligned}$$

Example: What is Var[X] when X is outcome of one fair die?

$$E[X^2] = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{6}\right) = \left(\frac{1}{6}\right)(91)$$

E[X] = 7/2, so

$$\text{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

properties of variance

In general:

$$\mathrm{Var}[X+Y] \neq \mathrm{Var}[X] + \mathrm{Var}[Y]$$

NOT linear

Ex 1:

Let $X = \pm 1$ based on 1 coin flip

As shown above, $E[X] = 0$, $\mathrm{Var}[X] = 1$

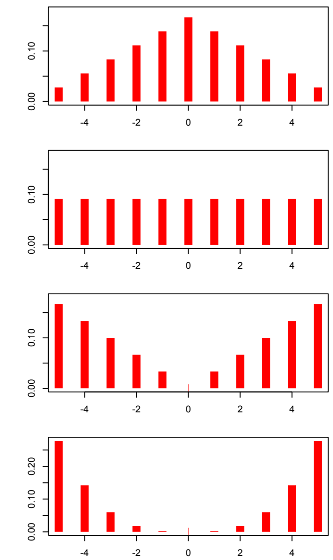
Let $Y = -X$; then $\mathrm{Var}[Y] = (-1)^2\mathrm{Var}[X] = 1$

But $X+Y = 0$, always, so $\mathrm{Var}[X+Y] = 0$

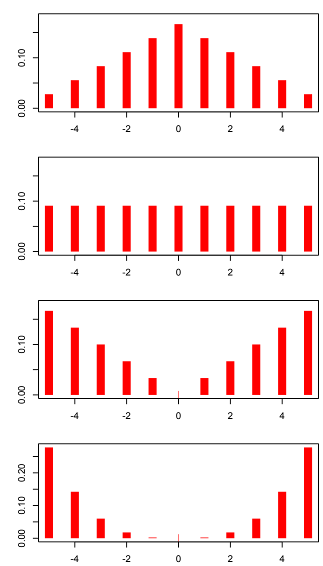
Ex 2:

As another example, is $\mathrm{Var}[X+X] = 2\mathrm{Var}[X]$?

more variance examples



more variance examples



$\sigma^2 = 5.83$

$\sigma^2 = 10$

$\sigma^2 = 15$

$\sigma^2 = 19.7$