more on expectation

For a discrete r.v. X with p.m.f. p(•), the *expectation of X*, aka *expected value* or *mean*, is

$\boxed{E[X] = \Sigma_x\, xp(x)}$ average of random values, *weighted* by their respective probabilities

For the equally-likely outcomes case, this is just the average of the possible random values of X

For *un*equally-likely outcomes, it is again the average of the possible random values of X, weighted by their respective probabilities

Ex 1: Let X = value seen rolling a fair die $p(1), p(2), ..., p(6) = 1/6$

$$E[X] = \sum_{i=1}^{6} ip(i) = \frac{1}{6}(1+2+\cdots+6) = \frac{21}{6} = 3.5$$

Ex 2: Coin flip; X = +1 if H (win \$1), -1 if T (lose \$1)

$E[X] = (+1)\cdot p(+1) + (-1)\cdot p(-1) = 1\cdot(1/2) + (-1)\cdot(1/2) = 0$

Two equivalent ways to compute expectation:

1. $$E(X) = \sum_{k \in Range(X)} kPr(X=k)$$
2. $$E(X) = \sum_{\omega \in S} X(w)Pr(w)$$

Example: X = number of people who get their homework back

Prob	outcome	X
1/6	123	3
1/6	132	1
1/6	213	1
1/6	231	0
1/6	312	0
1/6	321	1

Calculating $E[g(X)]$:

$Y=g(X)$ is a new r.v. Calculate $P[Y=j]$, then apply defn:

X = number of people who get their homework back

$Y = g(X) = X^2 \; mod\ 2$

$$E(g(X)) = \sum_{j \in Range(g(X))} jPr(g(X)=j)$$

$$= \sum_{k \in Range(X)} g(k)Pr(X=k)$$

Prob	outcome	X
1/6	123	3
1/6	132	1
1/6	213	1
1/6	231	0
1/6	312	0
1/6	321	1

Way 1

expectation of a *function* of a random variable

Calculating $E[g(X)]$:

$Y=g(X)$ is a new r.v. Calculate $P[Y=j]$, then apply defn:

X = sum of 2 dice rolls

i	$p(i) = P[X=i]$	$i \cdot p(i)$
2	1/36	2/36
3	2/36	6/36
4	3/36	12/36
5	4/36	20/36
6	5/36	30/36
7	6/36	42/36
8	5/36	40/36
9	4/36	36/36
10	3/36	30/36
11	2/36	22/36
12	1/36	12/36

$E[X] = \Sigma_i \, ip(i) =$ 252/36 $= 7$

$Y = g(X) = X \bmod 5$

j	$q(j) = P[Y = j]$	$j \cdot q(j)$
0	4/36+3/36 = 7/36	0/36
1	5/36+2/36 = 7/36	7/36
2	1/36+6/36+1/36 = 8/36	16/36
3	2/36+5/36 = 7/36	21/36
4	3/36+4/36 = 7/36	28/36

$E[Y] = \Sigma_j \, jq(j) =$ 72/36 $= 2$

Way 2

expectation of a *function* of a random variable

Calculating $E[g(X)]$: Another way – *add in a different order, using $P[X=...]$ instead of calculating $P[Y=...]$*

X = sum of 2 dice rolls

i	$p(i) = P[X=i]$	$g(i) \cdot p(i)$
2	1/36	2/36
3	2/36	6/36
4	3/36	12/36
5	4/36	0/36
6	5/36	5/36
7	6/36	12/36
8	5/36	15/36
9	4/36	16/36
10	3/36	0/36
11	2/36	2/36
12	1/36	2/36

$E[g(X)] = \Sigma_i \, g(i)p(i) =$ 72/36 $= 2$

$Y = g(X) = X \bmod 5$

j	$q(j) = P[Y = j]$	$j \cdot q(j)$
0	4/36+3/36 = 7/36	0/36
1	5/36+2/36 = 7/36	7/36
2	1/36+6/36+1/36 = 8/36	16/36
3	2/36+5/36 = 7/36	21/36
4	3/36+4/36 = 7/36	28/36

$E[Y] = \Sigma_j \, jq(j) =$ 72/36 $= 2$

properties of expectation

A & B each bet \$1, then flip 2 coins:

HH	A wins \$2
HT, TH	Each takes back \$1
TT	B wins \$2

Let X be A's net gain: +1, 0, -1, resp.:

$P(X = +1) = 1/4$
$P(X = 0) = 1/2$
$P(X = -1) = 1/4$

What is E[X]?

$E[X] = 1 \cdot 1/4 + 0 \cdot 1/2 + (-1) \cdot 1/4 = 0$

What is $E[X^2]$?

$E[X^2] = 1^2 \cdot 1/4 + 0^2 \cdot 1/2 + (-1)^2 \cdot 1/4 = 1/2$

Big Deal Note:
$E[X^2] \neq E[X]^2$