Counting - supplement

Generalized Product Rule

- If S is a set of sequences of length k for which there are
  - n_1 choices for the first element of sequence
  - n_2 choices for the second element given any particular choice for first
  - n_3 choices for third given any particular choice for first and second.
  - …..
- Then $|S| = n_1 \times n_2 \times .... \times n_k$

Example

10 people of different heights. How many ways to line up 5 of them?

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$$

Division Rule

- If f: A → B is k-to-1 function, then |A| = k|B|

Example:

- A is the set of ears in the room
- B is the set of people.
- Each ear maps to exactly one person.
- Each person has exactly two ears that map to it.
- Then the number of ears is twice # people

Combinations

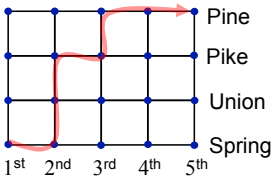
- Number of ways to choose r unordered objects out of n distinct objects
  - A: set of ordered lists of r out of n objects
  - B: set of unordered lists of r out of n objects
  - Each ordered list maps to one unordered list.
  - Each unordered list has r! ordered lists that map to it.
  - |A| = r! |B|

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1} = \frac{n!}{r!(n-r)!}$$

counting paths

How many ways to walk from 1st and Spring to 5th and Pine only going North and East?

Instead of tracing paths on the grid above, list choices. You walk 7 blocks; at each intersection choose N or E; must choose N exactly 3 times.

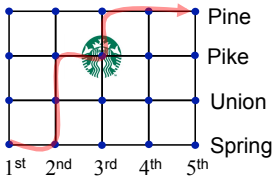


$$\binom{7}{3} = 35$$

counting paths

How many ways to walk from 1st and Spring to 5th and Pine only going North and East, if I want to stop at Starbucks on the way?

$$\binom{4}{2} \cdot \binom{3}{1}$$



Doughnuts

- You go to Top Pot to buy a dozen doughnuts. Your choices today are
  - Chocolate
  - Lemon-filled
  - Sugar
  - Glazed
  - Plain
- How many ways to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

Bijection Rule

- Count one set by counting another.
- Example:
  - A: all ways to select a dozen doughnuts when five varieties are available.
  - B: all 16 bit sequences with exactly 4 ones

$$\underbrace{00}_{\text{chocolate}}\ \underbrace{\ }_{\text{lemon-filled}}\ \underbrace{000000}_{\text{sugar}}\ \underbrace{00}_{\text{glazed}}\ \underbrace{00}_{\text{plain}}$$

Bijection between A and B

- A: all ways to select a dozen doughnuts when five varieties are available.
- B: all 16 bit sequences with exactly 4 ones

$$\underbrace{00}_{\text{chocolate}}\ \underbrace{\ }_{\text{lemon-filled}}\ \underbrace{000000}_{\text{sugar}}\ \underbrace{00}_{\text{glazed}}\ \underbrace{00}_{\text{plain}}$$

$$\underbrace{00}_{\text{chocolate}}\ 1\ \underbrace{\ }_{\text{lemon-filled}}\ 1\ \underbrace{000000}_{\text{sugar}}\ 1\ \underbrace{00}_{\text{glazed}}\ 1\ \underbrace{00}_{\text{plain}}$$

0011000000100100 .

$$\binom{16}{4}$$

Mapping from doughnuts to bit strings

c chocolate, l lemon-filled, s sugar, g glazed, and p plain

to the sequence:

$$\underbrace{0\ldots0}_{c}\ 1\ \underbrace{0\ldots0}_{l}\ 1\ \underbrace{0\ldots0}_{s}\ 1\ \underbrace{0\ldots0}_{g}\ 1\ \underbrace{0\ldots0}_{p}$$