CSE 312 Foundations II

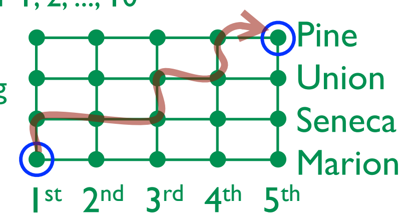
Counting

from slides by W.L. Ruzzo and others

How many ways are there to do X?

E.g., X = "choose an integer 1, 2, ..., 10"

E.g., X = "Walk from 1st & Marion to 5th & Pine, going only North or East at each intersection."



The Point:

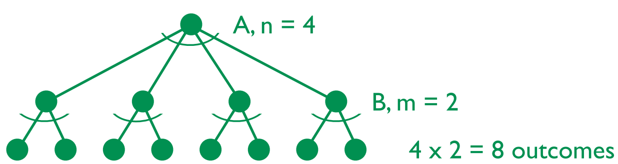
Counting gets hard when numbers are large, implicit and/or constraints are complex.

Systematic approaches help.

If there are

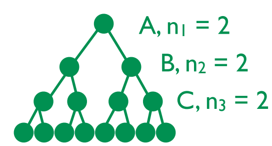
n outcomes/choices for some event A, sequentially followed by *m* outcomes/choices for event B,

then there are *n•m* outcomes/choices overall.



Q. How many n-bit numbers are there?

A. 1st bit 0 or 1, then 2nd bit 0 or 1, then ...



$$\overbrace{2 \cdot 2 \cdot \ldots \cdot 2}^{n} = 2^n$$

Q. How many subsets of a set of size n are there?

A. 1^{st} member in or out; 2^{nd} member in or out,... $\Rightarrow 2^n$

Tip: Visualize an order in which decisions are being made

Q. How many 4-character passwords are there, if each character must be one of a, b, ..., z, 0, 1, ..., 9 ?

A. 36 • 36 • 36 • 36 = 1,679,616 ≈ 1.7 million

Q. Ditto, but no character may be repeated?

A. 36 • 35 • 34 • 33 = 1,413,720 ≈ 1.4 million

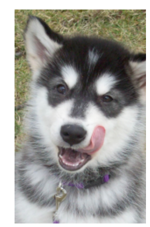
Q. How many arrangements of n distinct items are possible?

n	choices for 1st
(n-1)	choices for 2nd
(n-2)	choices for 3rd
...	...
1	choices for last

A. $\boxed{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n! \quad \text{(n factorial)}}$

Q. How many permutations of DAWGY are there?

A. 5! = 120



Q. How many of DAGGY?

A. 5!/2! = 60

$DAG_1G_2Y = DAG_2G_1Y$
$ADG_1YG_2 = ADG_2YG_1$
...

Q. How many of GODOGGY ?

A. $\frac{7!}{3!2!1!1!} = 420$

Q. Your elf-lord avatar can carry 3 objects chosen from

1. sword
2. knife
3. staff
4. water jug
5. iPad w/magic WiFi

How many ways can you equip him/her?

ordered ways in which to pick objects

$$\frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3! \cdot 2!} = 10$$

but picking abc is equiv to acb, and bca, and ...

Combinations: number of ways to choose r unordered things from n distinct things

"n choose r" aka binomial coefficients

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1} = \frac{n!}{r!(n-r)!}$$

Important special case:

how many (unordered) pairs from n objects

$$\binom{n}{2} = \frac{n(n-1)}{2} = \Theta(n^2)$$

Q. How many different poker hands are possible (i.e., 5 cards chosen from a deck of 52 distinct possibilities)?

A. $$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960$$

Q. 10 people meet at a party. If everyone shakes hands with everyone else, how many handshakes happen?

A. $$\binom{10}{2} = \frac{10 \cdot 9}{2 \cdot 1} = 45$$

Combinations: number of ways to choose r unordered things from n distinct things

"n choose r" aka binomial coefficients

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1} = \frac{n!}{r!(n-r)!}$$

Many Identities. E.g.:

$$\binom{n}{r} = \binom{n}{n-r}$$ ← by symmetry of definition

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$ ← first object either in or out; disjoint cases add

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$ ← by definition + algebra

Combinatorial argument:

Let S be a set of objects.

Show how to count the set one way —> N

Show how to count the set another way —> M

Conclude that N=M

$$\boxed{(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}}$$

proof 1: induction ...

proof 2: counting –

(x+y) • (x+y) • (x+y) • ... • (x+y)

pick either x or y from each factor

How many ways did you get exactly k x's? $\binom{n}{k}$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

coefficient of y^3 in $(7x+3y)^5$?

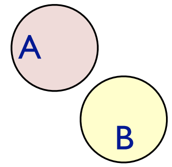
Relevant term: $\binom{5}{2}(7x)^2(3y)^3$

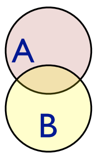
Coefficient: $\binom{5}{2}(7x)^2 3^3$

another general counting rule: inclusion-exclusion

If two sets or events A and B are *disjoint*, aka *mutually exclusive*, then

$|A \cup B| = |A| + |B|$



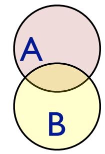


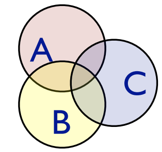
More generally, for two sets or events A and B, *whether or not they are disjoint,*

$|A \cup B| = |A| + |B| - |A \cap B|$

inclusion-exclusion

inclusion-exclusion in general





$|A \cup B| =$
$|A|+|B|-|A \cap B|$

$|A \cup B \cup C| =$
$|A| + |B| + |C|$
$- |B \cap C| - |A \cap C| - |A \cap B|$
$+ |A \cap B \cap C|$

General: + singles - pairs + triples - quads + ...

example

How many of $1, 2, \ldots, 10$ are divisible by 2, 3, and/or 5?

Let

$E_2 = \{x \mid 1 \le x \le 10 \wedge x \text{ is a multiple of } 2\}$

$E_3 = \{x \mid 1 \le x \le 10 \wedge x \text{ is a multiple of } 3\}$

$E_5 = \{x \mid 1 \le x \le 10 \wedge x \text{ is a multiple of } 5\}$

Notation: "AB" means "A and B"

$$\begin{aligned}
&|E_2 \cup E_3 \cup E_5| \\
&\quad = |E_2| + |E_3| + |E_5| - |E_2E_3| - |E_2E_5| - |E_3E_5| + |E_2E_3E_5| \\
&\quad = \left\lfloor \frac{10}{2} \right\rfloor + \left\lfloor \frac{10}{3} \right\rfloor + \left\lfloor \frac{10}{5} \right\rfloor - \left\lfloor \frac{10}{2 \cdot 3} \right\rfloor - \left\lfloor \frac{10}{2 \cdot 5} \right\rfloor - \left\lfloor \frac{10}{3 \cdot 5} \right\rfloor + \left\lfloor \frac{10}{2 \cdot 3 \cdot 5} \right\rfloor \\
&\quad = 5 + 3 + 2 - 1 - 1 - 0 + 0 \\
&\quad = 8
\end{aligned}$$

more counting: the pigeonhole principle

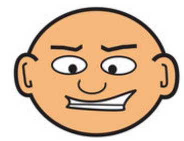
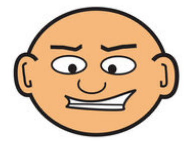
If there are n pigeons in k holes and n > k, then some hole contains more than one pigeon.
More precisely, some hole contains at least $\lceil n/k \rceil$ pigeons.

To solve a pigeonhole principle problem:

1. Define the pigeons
2. Define the pigeonholes
3. Define the mapping of pigeons to pigeonholes

If there are n pigeons in k holes and n > k, then some hole contains more than one pigeon.
More precisely, some hole contains at least $\lceil n/k \rceil$ pigeons.

There are two people in London who have the same number of hairs on their head.
Typical head ~ 150,000 hairs
Londoners have between 0 and 999,999 hairs on their head.
Since there are more than 1,000,000 people in London…

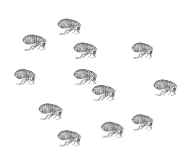
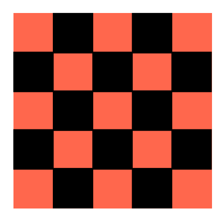
Pigeons: People in London > 1 million

i-th pigeonhole: i hairs on head (# pigeonholes 1 million)

pigeon —> pigeonhole: a person goes in i-th pigeonhole if that person has i hairs on his/her head.

Another example:

25 fleas sit on a 5 x 5 checkerboard, one per square. At the stroke of noon, all jump across an edge (not a corner) of their square to an adjacent square. Two must end up in the same square. Why?

Fleas on checkerboard

13 red squares, 12 black squares

Pigeons: fleas on red squares

Pigeonholes: black squares

Pigeon -> pigeonhole: red square flea maps to black square it jumps to.

summary

Product Rule: n_i outcomes for A_i: $\prod_i n_i$ in total (tree diagram)

Permutations:

ordered lists of n objects, no repeats: $n(n-1)\ldots 1 = n!$

ordered lists of r objects from n, no repeats: $n!/(n-r)!$

Combinations:

"n choose r," aka binomial coefficients,

unordered lists of r objects from n $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Binomial Theorem: $(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$

Inclusion-Exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$

Pigeonhole Principle