Recipe for finding pdf of $g(X)$

$Y = g(X)$

1) Find CDF

$$F_Y(y) = Pr(g(X) \le y) = \int_{\{x \mid g(x) \le y\}} f_X(x)\,dx$$

2) Differentiate to get pdf

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$X \sim U[0,1]$ What is pdf of $Y = e^X$?

Note $X > 0$ in $[0,1]$ $\Rightarrow Y > 0$ in $[1, e]$

$$\Pr(Y \le y) = \Pr(e^X \le y)$$
$$= \Pr(X \le \ln(y))$$
$$= \int_0^{\ln y} f(x)\,dx = \ln(y)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{y} \qquad 1 \le y \le e$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_1^e y \cdot \frac{1}{y}\,dy = e - 1$$

$$E(Y) = \int_{-\infty}^{\infty} e^x f_X(x)\,dx = \int_0^1 e^x\,dx = e - 1$$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

X cont w/ CDF F_X pmf f_X

find density fn of $Y = 2X$

$$F_Y(y) = Pr(Y \le y)$$
$$= Pr(2X \le y)$$
$$= Pr\left(X \le \frac{y}{2}\right)$$
$$= F_X\left(\frac{y}{2}\right)$$

$$\frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y}{2}\right) = \frac{1}{2} f_X\left(\frac{y}{2}\right)$$

To see that this makes sense

$$Pr\left(a - \frac{\varepsilon}{2} \le X \le a + \frac{\varepsilon}{2}\right) \approx \varepsilon f(a)$$

$$\varepsilon f_Y(a) \approx Pr\left(a - \frac{\varepsilon}{2} \le Y \le a + \frac{\varepsilon}{2}\right)$$
$$= Pr\left(a - \frac{\varepsilon}{2} \le 2X \le a + \frac{\varepsilon}{2}\right)$$
$$= Pr\left(\frac{a}{2} - \frac{\varepsilon}{4} \le X \le \frac{a}{2} + \frac{\varepsilon}{4}\right)$$
$$\approx \frac{\varepsilon}{2} f_X\left(\frac{a}{2}\right) = \varepsilon \cdot \frac{1}{2} f_X\left(\frac{a}{2}\right)$$

$X_1, \dots, X_n$ iid. $U[0,1]$ What is $E(\max(X_1, .., X_n))$?

Let $X = \max(X_1, .., X_n)$

$$F_X(x) = Pr(\max(X_1, .., X_n) \le x) = Pr(X_1 \le x, X_2 \le x, \dots, X_n \le x)$$

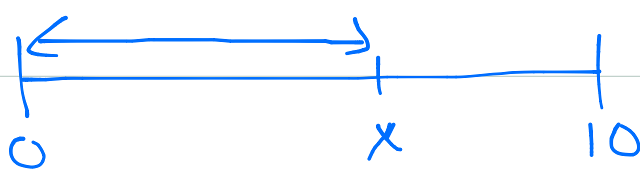
$$= x^n$$

$$f_X(x) = \frac{d}{dx} F_X(x) = n x^{n-1}$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x \cdot n x^{n-1} dx = n \int_0^1 x^n dx = n \frac{x^{n+1}}{n+1} \Big|_0^1 = \frac{n}{n+1}$$

$X \sim Unif[0..10]$

$Y \sim Unif[0..X]$



$E(Y) = ?$

$$E(Y) = \int_{-\infty}^{\infty} E(Y | X = u) f_X(u) du$$

$Y \mid X=u \sim \text{Unif}[0..u]$

$$E(Y) = \int_0^{10} \frac{u}{2} \frac{1}{10}\,du = \frac{u^2}{40}\Big|_0^{10} = \frac{100}{40} = 2.5$$

$$\text{or } \frac{1}{2} \int_0^{10} u \frac{1}{10}\,du = \frac{5}{2} = 2.5$$

$f_{XY}(x,y)$? $f_Y(y)$?

$$f_{X,Y}(x,y) = f_X(x)\, f_{Y|X=x}(y)$$

$$= \frac{1}{10} \cdot \frac{1}{x} \qquad 0 \le y \le x \le 10$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_y^{10} \frac{1}{10x}\,dx = \frac{1}{10} \ln x \Big|_y^{10} = \frac{1}{10} \ln\left(\frac{10}{y}\right) \qquad 0 < y \le 10$$

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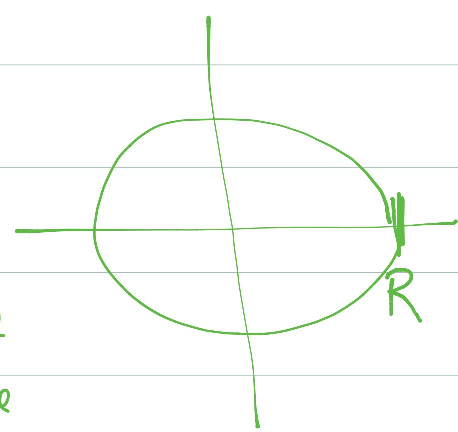
circle of radius R

centered at origin

(X,Y) coordinates of random pt in circle

("dart" equally likely to fall anywhere



$$\Rightarrow f(x,y) = \begin{cases} c & x^2+y^2 \le R \\ 0 & \text{o.w.} \end{cases}$$

① What is c?

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$c \iint_{x^2+y^2\le R} dx\,dy = 1$$

use polar coordinates or observe $\iint$ = area of circle

$$\Rightarrow c = \frac{1}{\pi R^2}$$

② What is marginal density of X?

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{\pi R^2} \int_{x^2+y^2\le R} dy$$

$$= \frac{1}{\pi R^2} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \frac{2}{\pi R^2}\sqrt{R^2-x^2} \qquad x^2 \le R^2$$

$$0 \quad \text{o.w.}$$

③ Pr(distance from origin to pt selected $\le a$) $\qquad D = \sqrt{X^2+Y^2}$

$$F_D(a) = \Pr\left(\sqrt{X^2+Y^2} \le a^2\right)$$

$$= \Pr\left(X^2+Y^2 \le a^2\right)$$

$$= \int\limits_{x^2+y^2 \le a^2} f(x,y)\,dy\,dx$$

$$= \frac{1}{\pi R^2} \underbrace{\int\limits_{x^2+y^2 \le a^2} dy\,dx}_{\text{area of circle of radius } a} \doteq \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}$$

④ $E(D)$

$f_D(a) = \frac{d}{da} F_D(a) = \frac{2a}{R^2}$ for $0 \le a \le R$
0 o.w.

$$E(D) = \int_0^R a f_D(a) = \frac{2}{R^2} \int_0^R a^2\,da = \frac{2}{3} R$$

$f(x,y) = x e^{-x(y+1)}$ $x>0,\ y>0$

What is $f(X \mid Y=y) = \frac{f(x,y)}{f_Y(y)}$? $f_Y(y) = \int_0^\infty x e^{-x(y+1)}\,dx$

Find den ity of. $Z = XY$

$F_Z(z) = Pr(XY \le z)$

$$= \int_0^\infty \Pr(XY \leq z \mid X = x)\, f_X(x)\, dx$$

$$= \int_0^\infty \Pr\left(Y \leq \frac{z}{x} \,\middle|\, X = x\right) f_X(x)\, dx$$

$$= \int_0^\infty F_Y\left(\frac{z}{x}\right) f_X(x)\, dx$$