

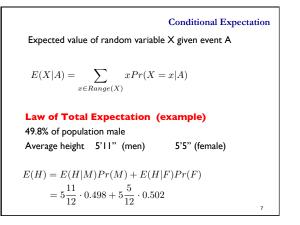
$$\begin{aligned} & \text{Conditional distributions} \\ & \text{Let X and Y be discrete r.v.s.} \\ & \text{Conditional probability mass function of X given that Y=y} \\ & p_{X|Y}(x|y) = Pr(X = x|Y = y) \\ & = \frac{Pr(X = x, Y = y)}{Pr(Y = y)} = \frac{p(x, y)}{p_Y(y)} \\ & \sum_x p_{X|Y}(x|y) = ? \\ & \sum_x p_{X|Y}(x|y) = \sum_x \frac{p(x, y)}{p_Y(y)} = \frac{p_Y(y)}{p_Y(y)} = 1 \end{aligned}$$

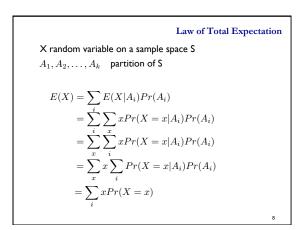
$$\begin{aligned} & \text{Conditional distributions} \\ & p_{X|Y}(x|y) = Pr(X=x|Y=y) \end{aligned}$$
 If X and Y are independent Poisson random variables with respective parameters  $\lambda_1$  and  $\lambda_2$ , calculate the conditional distribution of X, given that X + Y = n. \end{aligned} 
$$P(X=k|X+Y=n) = \frac{P(X=k,X+Y=n)}{P(X+Y=n)} \\ &= \frac{P(X=k,Y=n-k)}{P(X+Y=n)} \\ &= \frac{P(X=k)P(Y=n-k)}{P(X+Y=n)} \end{aligned}$$

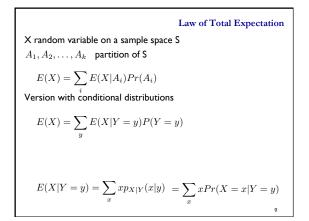
Sum of Poisson random variables is Poisson  

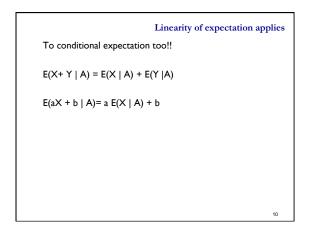
$$X \sim Poi(\lambda_1) \qquad Y \sim Poi(\lambda_2)$$
Show that  $X + Y \sim Poi(\lambda_1 + \lambda_2)$ 
Proof:  $P(X + Y = n) = \sum_{k=0}^{n} P(X = k, Y = n - k)$   
 $= \sum_{k=0}^{n} P(X = k)P(Y = n - k)$   
 $= \sum_{k=0}^{n} e^{-\lambda_k} \frac{\lambda_k^k}{k!} \cdot e^{-\lambda_2} \frac{\lambda_k^{n-k}}{(n-k)!}$   
 $= e^{-(\lambda_1 + \lambda_2)} \sum_{k=0}^{n} \frac{\lambda_k^k \lambda_k^{n-k}}{k!(n-k)!}$   
 $= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda_k^k \lambda_2^{n-k}$   
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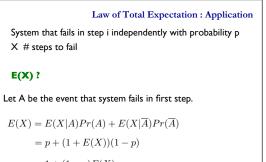
$$\begin{aligned} & \text{Conditional distributions} \\ & \text{If X and Y are independent Poisson random variables with} \\ & \text{respective parameters } \lambda_1 \text{ and } \lambda_2, \text{ calculate the conditional} \\ & \text{distribution of X, given that X + Y = n.} \\ & \frac{P(X=k)P(Y=n-k)}{P(X+Y=n)} = \frac{e^{-\lambda_1 \frac{\lambda_1^k}{k!}}e^{-\lambda_2 \frac{\lambda_2^{n-k}}{(n-k)!}}}{e^{-(\lambda_1+\lambda_2) \frac{(\lambda_1+\lambda_2)^n}{n!}}} \\ & = \binom{n}{k} \frac{\lambda_1^k}{(\lambda_1+\lambda_2)^k} \cdot \frac{\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^{n-k}} \\ & \text{The conditional distribution of X, given that X+Y=n is:} \\ & \text{Binomial (n, } \frac{\lambda_1}{(\lambda_1+\lambda_2)} ) \end{aligned}$$











$$= 1 + (1 - p)E(X)$$
$$E(X) = \frac{1}{p}$$

Law of Total Expectation : Example
A miner is trapped in a mine containing 3 doors.
D<sub>1</sub>: The 1<sup>st</sup> door leads to a tunnel that will take him to safety after 3 hours.
D<sub>2</sub>: The 2<sup>nd</sup> door leads to a tunnel that returns him to the mine after 5 hours.
D<sub>3</sub>: The 3<sup>rd</sup> door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters (12, 1/3).
At all times, he is equally likely to choose any one of the doors.
E(time to reach safety) ?

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## Law of Total Expectation : Example A miner is trapped in a mine containing 3 doors. D<sub>1</sub>: The 1<sup>st</sup> door leads to a tunnel that will take him to safety after 3 hours. D<sub>2</sub>: The 2<sup>nd</sup> door leads to a tunnel that returns him to the mine after 5 hours. D<sub>3</sub>: The 3<sup>rd</sup> door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters (12, 1/3). At all times, he is equally likely to choose any one of the doors. E(time to reach safety) ?

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$$E(T) = E(T|D_1)\frac{1}{3} + E(T|D_2)\frac{1}{3} + E(T|D_3)\frac{1}{3}$$
  
=  $3 \cdot \frac{1}{3} + 5 \cdot \frac{1}{3} + (4 + E(T)) \cdot \frac{1}{3}$   
 $E(T) = 6$ 

## Problem

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers.

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X number of people who enter  
Y number of stops
$$Pr(X = k) = e^{-10} \frac{10^k}{k!}$$

$$E(Y) = \sum_{k=0}^{\infty} E(Y|X = k)P(X = k)$$

$$E(Y|X = k) = E(Y_1 + \dots + Y_N|X = k)$$

$$Y_i \text{ indicates a stop on floor i}$$

$$E(Y_i|X = k) = \left(1 - (1 - 1/N)^k\right)$$
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## Game of Craps

- Begin by rolling an ordinary pair of dice
- If the sum of dice is 2, 3 or 12, the player loses
- If the sum of dice is 7 or 11, the player wins
- If it is any other number, say k, the player continues to roll the dice until the sum is either 7 or k.
  - If it is 7, the player loses.
  - If it is k, the player wins.

Let R denote the number of rolls of the dice in a game of craps.

• What is E(R)?