Conditional distributions, conditional expectation and the law of total expectation.

Conditional distributions

Let X and Y be discrete r.v.s.

Conditional probability mass function of X given that Y=y

$$p_{X|Y}(x|y) = Pr(X = x|Y = y)$$
$$= \frac{Pr(X = x, Y = y)}{Pr(Y = y)} = \frac{p(x,y)}{p_Y(y)}$$

$$\sum_x p_{X|Y}(x|y) = ?$$

$$\sum_x p_{X|Y}(x|y) = \sum_x \frac{p(x,y)}{p_Y(y)} = \frac{p_Y(y)}{p_Y(y)} = 1$$

Conditional distributions

$$p_{X|Y}(x|y) = Pr(X = x|Y = y)$$

If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2 , calculate the conditional distribution of X, given that X + Y = n.

$$P(X = k|X + Y = n) = \frac{P(X = k, X + Y = n)}{P(X + Y = n)}$$
$$= \frac{P(X = k, Y = n - k)}{P(X + Y = n)}$$
$$= \frac{P(X = k)P(Y = n - k)}{P(X + Y = n)}$$

Sum of Poisson random variables is Poisson

$$X \sim Poi(\lambda_1) \qquad Y \sim Poi(\lambda_2)$$

Show that $X + Y \sim Poi(\lambda_1 + \lambda_2)$

Proof:
$$P(X+Y=n) = \sum_{k=0}^{n} P(X=k, Y=n-k)$$
$$= \sum_{k=0}^{n} P(X=k)P(Y=n-k)$$
$$= \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}$$
$$= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!}$$
$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\lambda_1^k \lambda_2^{n-k}$$
$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n$$

Conditional distributions

If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X, given that X + Y = n.

$$\frac{P(X = k)P(Y = n - k)}{P(X + Y = n)} = \frac{e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \binom{n}{k}\frac{\lambda_1^k}{(\lambda_1+\lambda_2)^k} \cdot \frac{\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^{n-k}}$$

The conditional distribution of X, given that X+Y=n is:
Binomial (n, $\frac{\lambda_1}{(\lambda_1+\lambda_2)}$)

Conditional Expectation

Expected value of random variable X given event A

$$E(X|A) = \sum_{x \in Range(X)} xPr(X = x|A)$$

Law of Total Expectation (example)

49.8% of population male

Average height 5'11" (men) 5'5" (female)

$$E(H) = E(H|M)Pr(M) + E(H|F)Pr(F)$$
$$= 5\frac{11}{12} \cdot 0.498 + 5\frac{5}{12} \cdot 0.502$$

Law of Total Expectation

X random variable on a sample space S

$A_1, A_2, \ldots, A_k$ partition of S

$$\begin{aligned} E(X) &= \sum_i E(X|A_i)Pr(A_i) \\ &= \sum_i \sum_x xPr(X=x|A_i)Pr(A_i) \\ &= \sum_x \sum_i xPr(X=x|A_i)Pr(A_i) \\ &= \sum_x x \sum_i Pr(X=x|A_i)Pr(A_i) \\ &= \sum_i xPr(X=x) \end{aligned}$$

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Law of Total Expectation

X random variable on a sample space S

$A_1, A_2, \ldots, A_k$ partition of S

$$E(X) = \sum_i E(X|A_i)Pr(A_i)$$

Version with conditional distributions

$$E(X) = \sum_y E(X|Y=y)P(Y=y)$$

$$E(X|Y=y) = \sum_x xp_{X|Y}(x|y) \quad = \sum_x xPr(X=x|Y=y)$$

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Linearity of expectation applies

To conditional expectation too!!

E(X+ Y | A) = E(X | A) + E(Y |A)

E(aX + b | A)= a E(X | A) + b

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Law of Total Expectation : Application

System that fails in step i independently with probability p

X # steps to fail

E(X) ?

Let A be the event that system fails in first step.

$$\begin{aligned} E(X) &= E(X|A)Pr(A) + E(X|\overline{A})Pr(\overline{A}) \\ &= p + (1 + E(X))(1-p) \\ &= 1 + (1-p)E(X) \end{aligned}$$

$$E(X) = \frac{1}{p}$$

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Law of Total Expectation : Example

A miner is trapped in a mine containing 3 doors.

- D_1: The 1st door leads to a tunnel that will take him to safety after 3 hours.
- D_2: The 2nd door leads to a tunnel that returns him to the mine after 5 hours.
- D_3: The 3rd door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters (12, 1/3).

At all times, he is equally likely to choose any one of the doors.

E(time to reach safety) ?

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E(time to reach safety) ?

$$\begin{aligned} E(T) &= E(T|D_1)\frac{1}{3} + E(T|D_2)\frac{1}{3} + E(T|D_3)\frac{1}{3} \\ &= 3 \cdot \frac{1}{3} + 5 \cdot \frac{1}{3} + (4 + E(T)) \cdot \frac{1}{3} \end{aligned}$$

$$E(T) = 6$$

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Problem

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers.

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X number of people who enter

Y number of stops

$$Pr(X=k) = e^{-10}\frac{10^k}{k!}$$

$$E(Y) = \sum_{k=0}^{\infty} E(Y|X=k)P(X=k)$$

$$E(Y|X=k) = E(Y_1 + \ldots + Y_N|X=k)$$

Y_i indicates a stop on floor i

$$E(Y_i|X=k) = \left(1-(1-1/N)^k\right)$$

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Game of Craps

- Begin by rolling an ordinary pair of dice
- If the sum of dice is 2, 3 or 12, the player loses
- If the sum of dice is 7 or 11, the player wins
- If it is any other number, say k, the player continues to roll the dice until the sum is either 7 or k.
  - If it is 7, the player loses.
  - If it is k, the player wins.

Let R denote the number of rolls of the dice in a game of craps.

- What is E(R)?

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