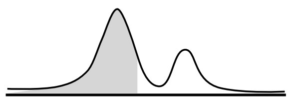
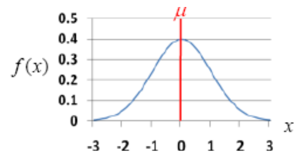
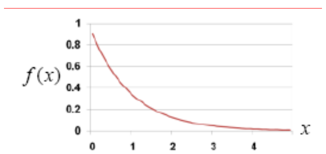
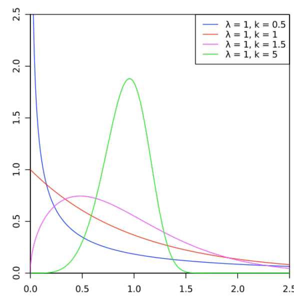




limit theorems





Limit Theorems

Consider i.i.d. (independent, identically distributed) random vars $X_1, X_2, X_3, \ldots$

X_i has $\mu = E[X_i]$ and $\sigma^2 = Var[X_i]$

Consider random variables

$$X_1 + X_2 + \ldots + X_n$$

and

$$\frac{1}{n}\sum_{i=1}^{n} X_i$$

Law of Large Numbers

If we observe a random variable X many times (independently) and take the average, this average will converge to a real number which is E(X).

Formally, let $X_1, \ldots, X_n$ be independent, identically distributed random variables with mean μ.

Define $A_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ Then for any $\alpha > 0$ we have

$$Pr(|A_n - \mu| > \alpha) \to 0 \quad \text{as} \quad n \to \infty$$

Proof: Use Chebychev's inequality.

the central limit theorem (CLT)

Consider i.i.d. (independent, identically distributed) random vars $X_1, X_2, X_3, \ldots$

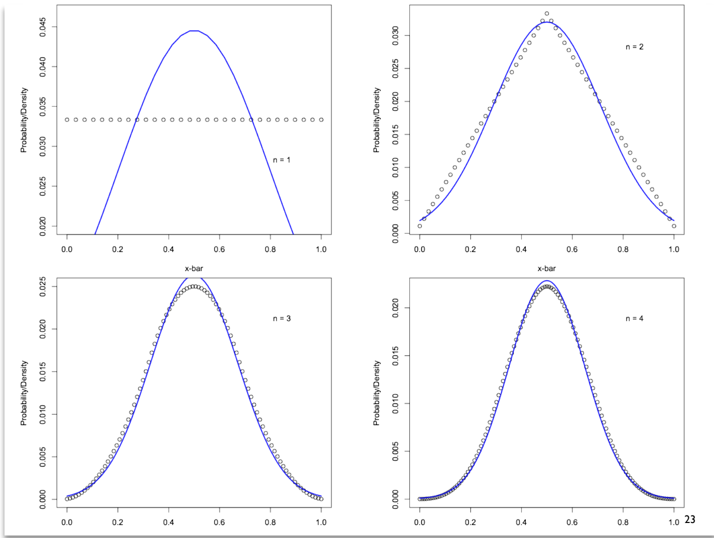
X_i has $\mu = E[X_i]$ and $\sigma^2 = Var[X_i]$

As $n \to \infty$,

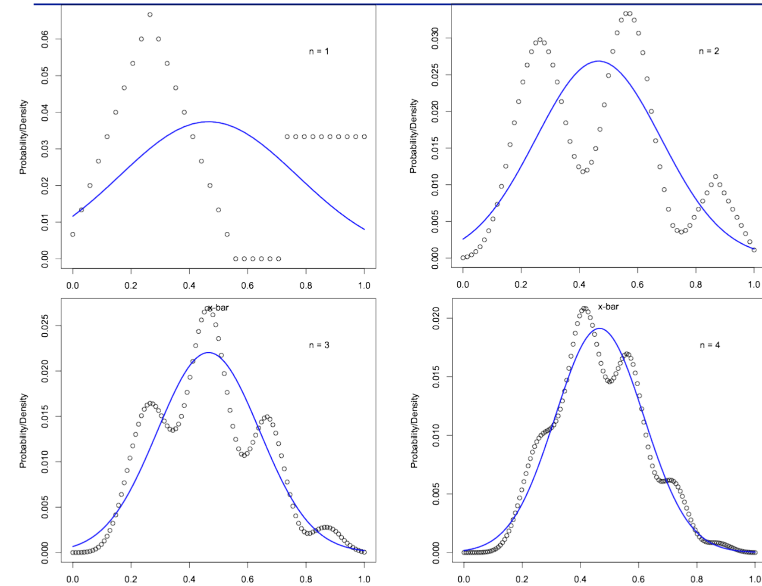
$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \longrightarrow N(0,1)$$

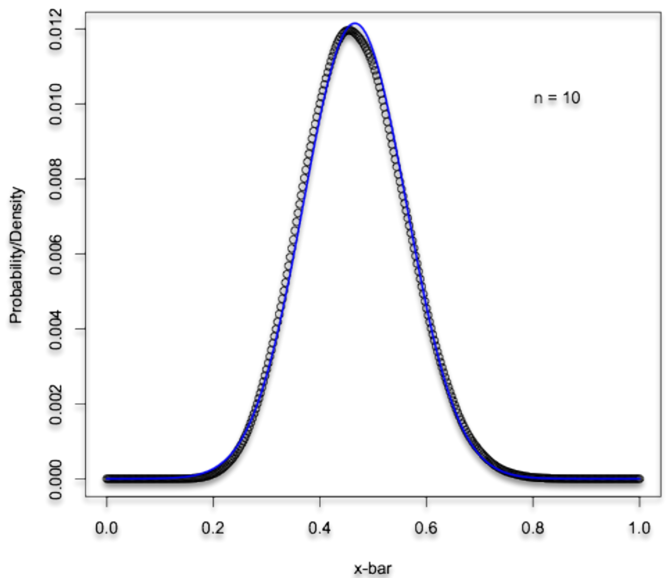
Restated: As $n \to \infty$,

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to N\left(\mu, \frac{\sigma^2}{n}\right)$$



CLT applies even to even wacky distributions



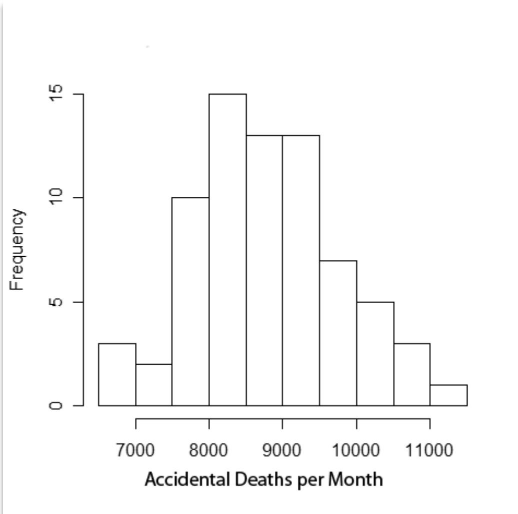


CLT in the real world

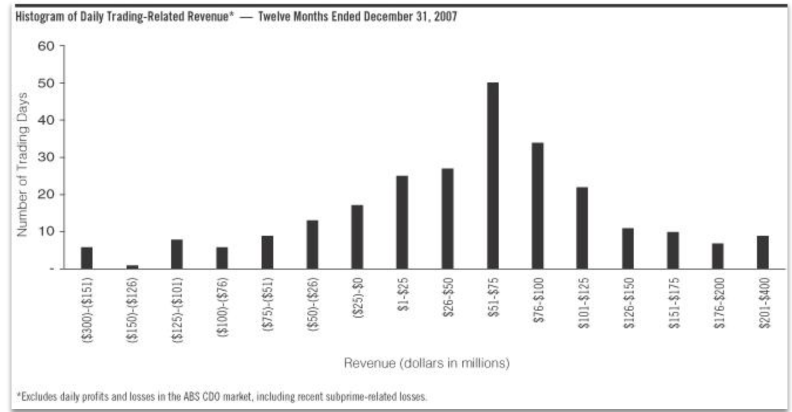
CLT is the reason many things appear normally distributed
Many quantities = sums of (roughly) independent random vars

Exam scores: sums of individual problems
People's heights: sum of many genetic & environmental factors
Measurements: sums of various small instrument errors
...

in the real world…



in the real world…



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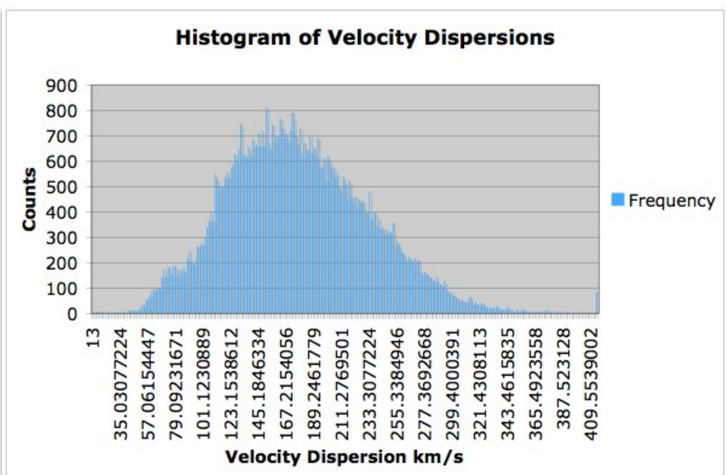

the central limit theorem (CLT)

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Example 1

Number of students who enroll in a class is Poisson (100).

Professor will teach course in two sections if more than 120 students enroll. What is the probability of two sections?

When applying approximation, use continuity correction:

Think of Pr (X=i) = Pr (i-0.5 < X < i + 0.5)

Recall Poisson (100) is sum of 100 independent Poisson(1) random variables, so we can apply CLT.

$$Pr(X > 119.5) = Pr\left(\frac{X - 100}{\sqrt{100}} \geq \frac{119.5 - 100}{\sqrt{100}}\right)$$

$$\approx 1 - \Phi(1.95)$$

$$\approx 0.0256.$$

Example 2

If 10 fair die are rolled, find the approximate probability that the sum obtained is between 30 and 40, inclusive, using CLT.

X_i is the value of die # i.

$E(X_i)$ = 7/2 and $Var(X_i)$ = 35/12

$$E(X) = 35$$

$$X = X_1 + X_2 + ...X_{10} \qquad Var(X) = 350/12$$

$$Pr(29.5 \leq X \leq 40.5) = Pr\left(\frac{29.5 - 35}{\sqrt{350/12}} \leq \frac{X - 35}{\sqrt{350/12}} \leq \frac{40.5 - 35}{\sqrt{350/12}}\right)$$

$$\approx 2 \cdot \Phi(1.0184) - 1$$

$$\approx 0.692.$$