Oct. 7, 2016

Equally Probable Outcomes

365 birthdays, n people

P(no shared birthday)

$$\boxed{|\Omega| = 365^n}$$

$$~~|E| = \binom{365}{n}~~ ?$$

$$|E| = \frac{365!}{(365-n)!} \quad \cancel{?}$$

$$P = \frac{|E|}{|\Omega|}$$

$$\underline{E \subseteq \Omega}$$

Ex: n chips manufactured, d of which are defective. k chips are selected randomly for testing. What is P(k selected chips contain some defectives)? Let E be event that none of the k chips is defective.

$$P(E) = \frac{|E|}{|\Omega|} = \frac{\binom{n-d}{k}}{\binom{n}{k}} \quad \text{so}$$

$$P(\overline{E}) = 1 - P(E) = 1 - \frac{\binom{n-d}{k}}{\binom{n}{k}}$$

Defn: The conditional probability of E given F, written $P(E|F)$ is the probability of E, given that F was observed. Sample space is reduced to F and event is reduced to $E \cap F$.

In the case of equally probable outcomes,

$$P(E|F) = \frac{|E \cap F|}{|F|}$$

$$= \frac{|E \cap F| / |\Omega|}{|F| / |\Omega|} = \frac{P(E \cap F)}{P(F)}$$

Ex: Roll a fair 6-sided die.

$P(3 \mid \text{odd})$?

$E = \{3\}$, $F = \{1,3,5\}$, $\Omega = \{1,2,3,4,5,6\}$

$$P(3 \mid \text{odd}) = \frac{|E \cap F|}{|F|} = \frac{|E|}{|F|} = \frac{1}{3}$$

$$P(3 \mid \text{odd}) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} = \frac{1/6}{1/2} = \frac{1}{3}$$

Ex: Let Ω be all ways to dealing two 5-card hands.
Let Y be the event that you are dealt no spade.
Let Θ " " " your opponent is dealt no spade.

$$P(\Theta \mid Y) = \frac{|\Theta \cap Y|}{|Y|} = \frac{\binom{39}{5}\binom{34}{5}}{\binom{39}{5}\binom{47}{5}} \approx 0.18$$

Compare to $P(\Theta) \approx 0.22$

$$P(\text{Opp dealt} \geq 1 \text{ spade} \mid Y) = 1 - P(\Theta \mid Y) \approx 0.82$$

Outcomes not equally probable.

Ex: Let $\Omega = \{0, 1, 2\}$, the number of heads in 2 coin flips of a fair coin. $P(0) = P(2) = \frac{1}{4}$, but $P(1) = \frac{1}{2}$.

Defn: $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}$, assuming $P(F) \neq 0$.

Ex: Let E be event "2 heads" and F be event "≥ 1 head".

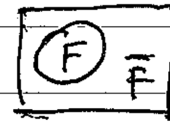
$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} = \frac{1/4}{1/2 + 1/4} = \frac{1}{3}$$

Chain rule: $P(E \cap F) = P(F)\,P(E \mid F)$

$P(E_1 \cap E_2 \cap \cdots \cap E_n) =$

$$P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1, E_2) \cdots P(E_n \mid E_1, E_2, \ldots, E_{n-1})$$

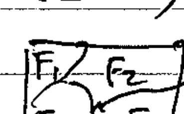
Law of Total Probability: If E and F are events,

$$P(E) = P((E \cap F) \cup (E \cap \bar{F}))$$  Ω

$$= P(E \cap F) + P(E \cap \bar{F})$$

$$= P(E|F)P(F) + P(E|\bar{F})P(\bar{F}) \leftarrow$$

More generally, If $F_1, F_2, \cdots F_n$ partition Ω,

$$P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$$ 

Ω

Ex: Sally will take either chem or physics.
She will get an A in physics with prob 3/4 and an A in chem with prob 3/5.
She flips a fair coin to decide which to take.

$$P(A) = P(A|\text{Phys})P(\text{Phys}) + P(A|\text{Chem})P(\text{Chem})$$

$$= \frac{3}{4} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} = \frac{27}{40}$$

Gambler's Ruin

A has \$$i$, B has \$$(N-i)$.

Flip a fair coin: $H \to$ A wins \$1 from B,
$T \to$ B wins \$1 from A.

Whoever gets all N dollars wins.