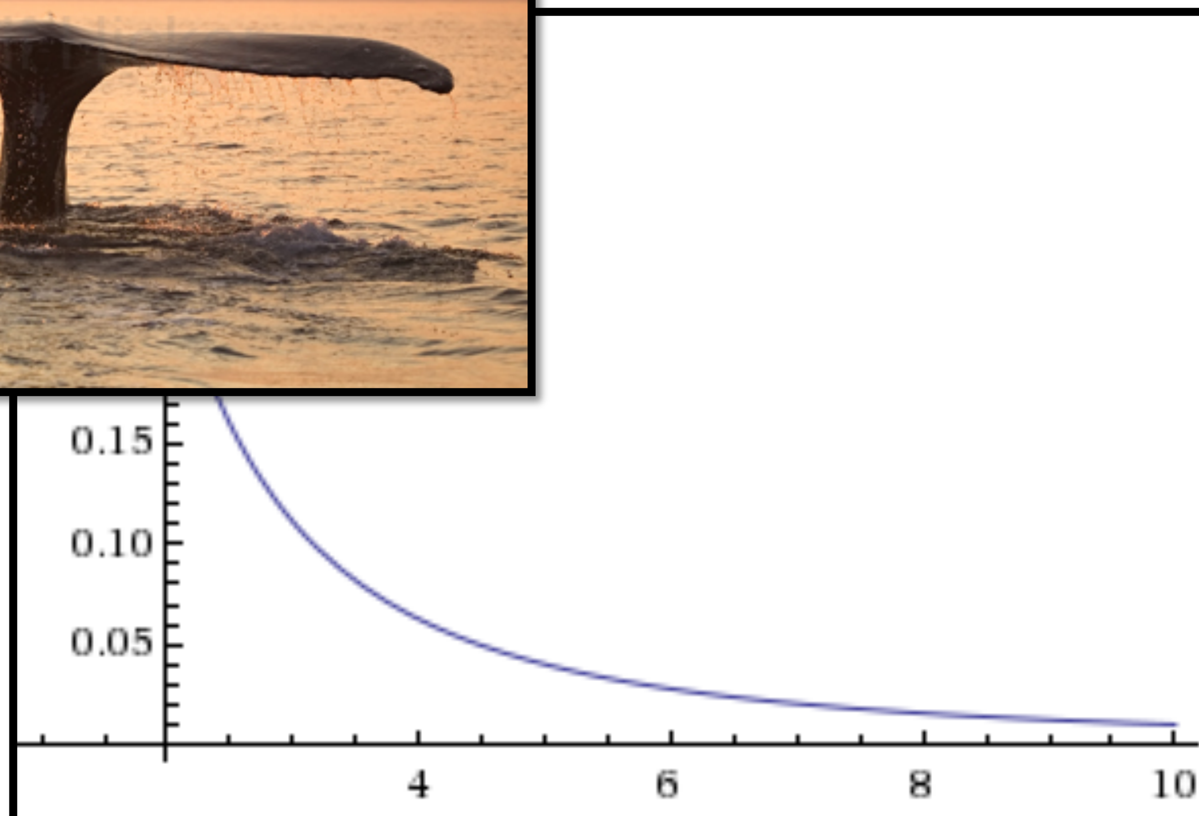
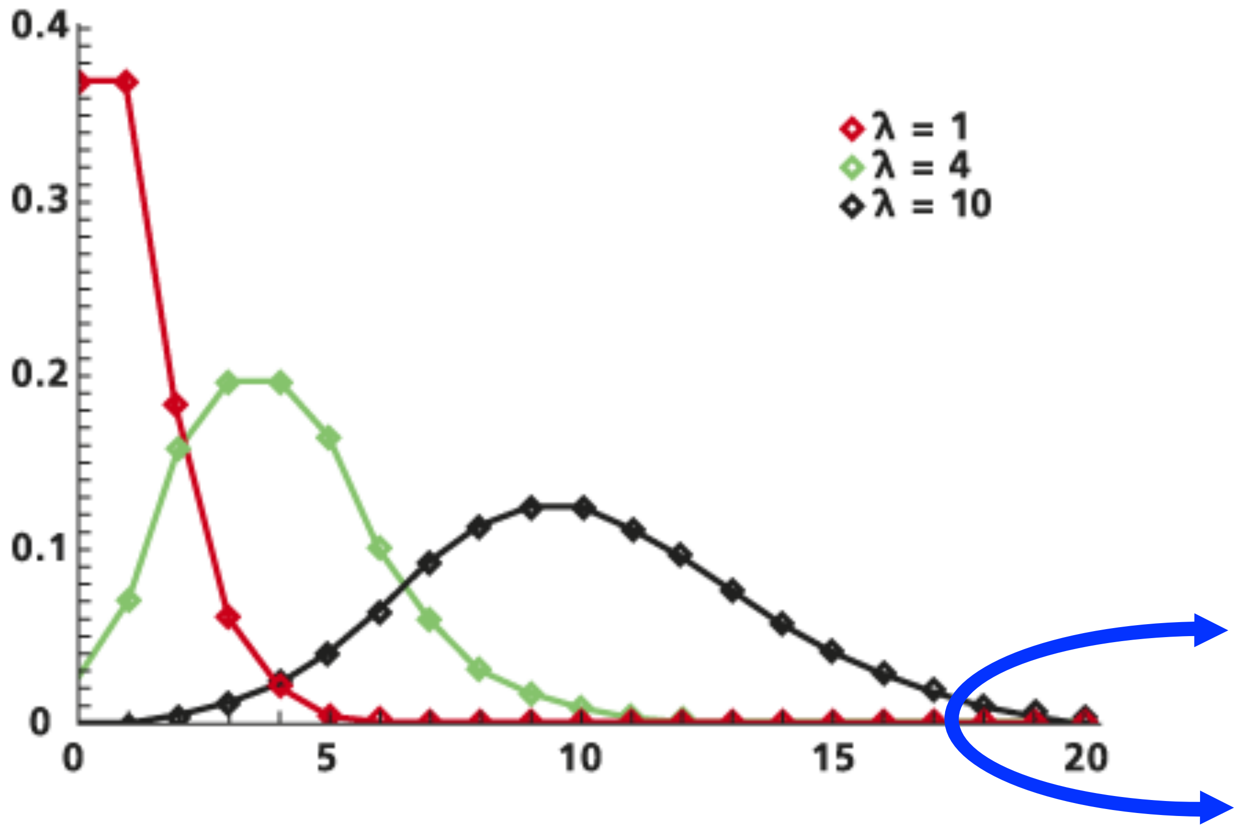
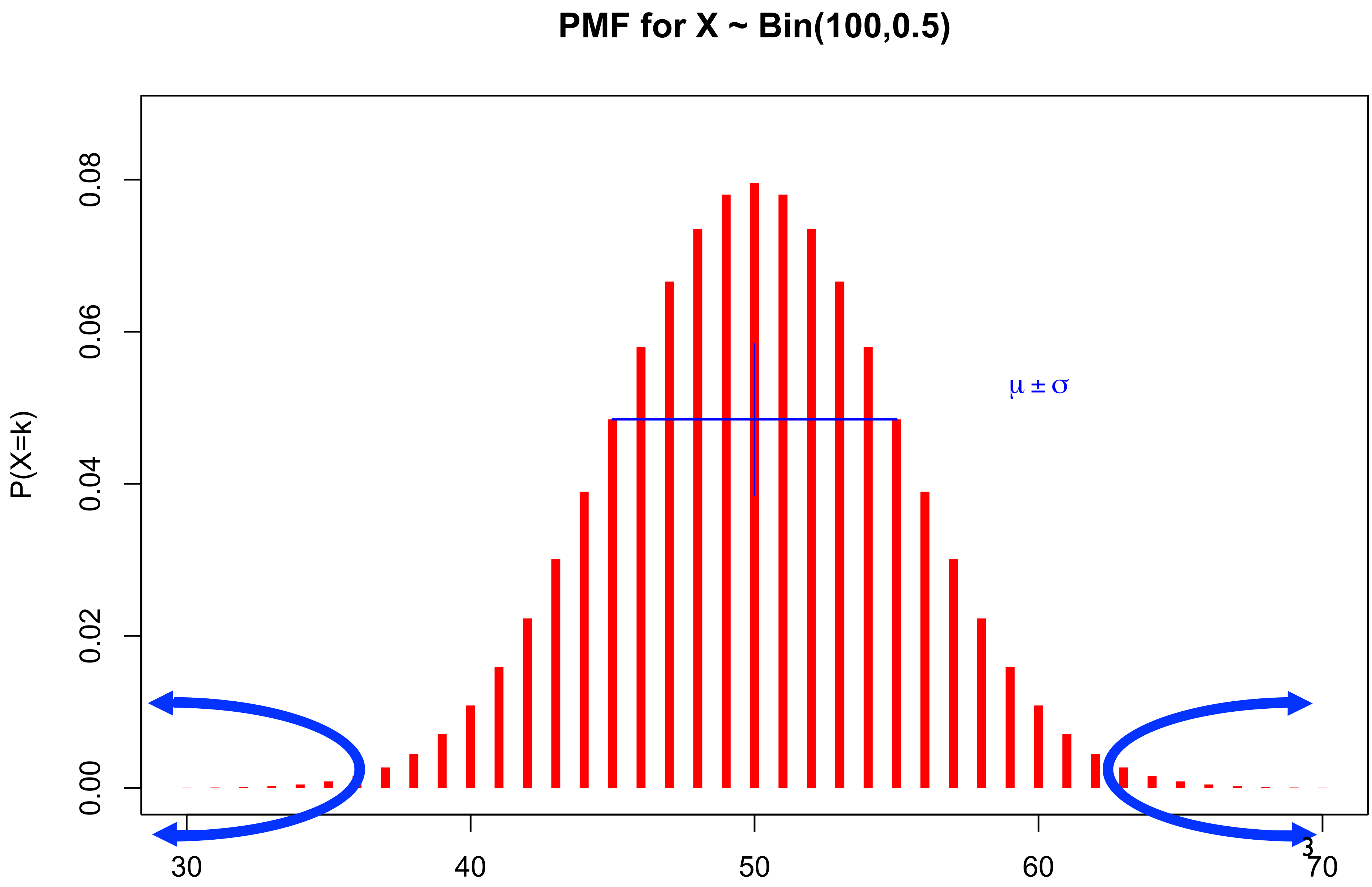
tail bounds



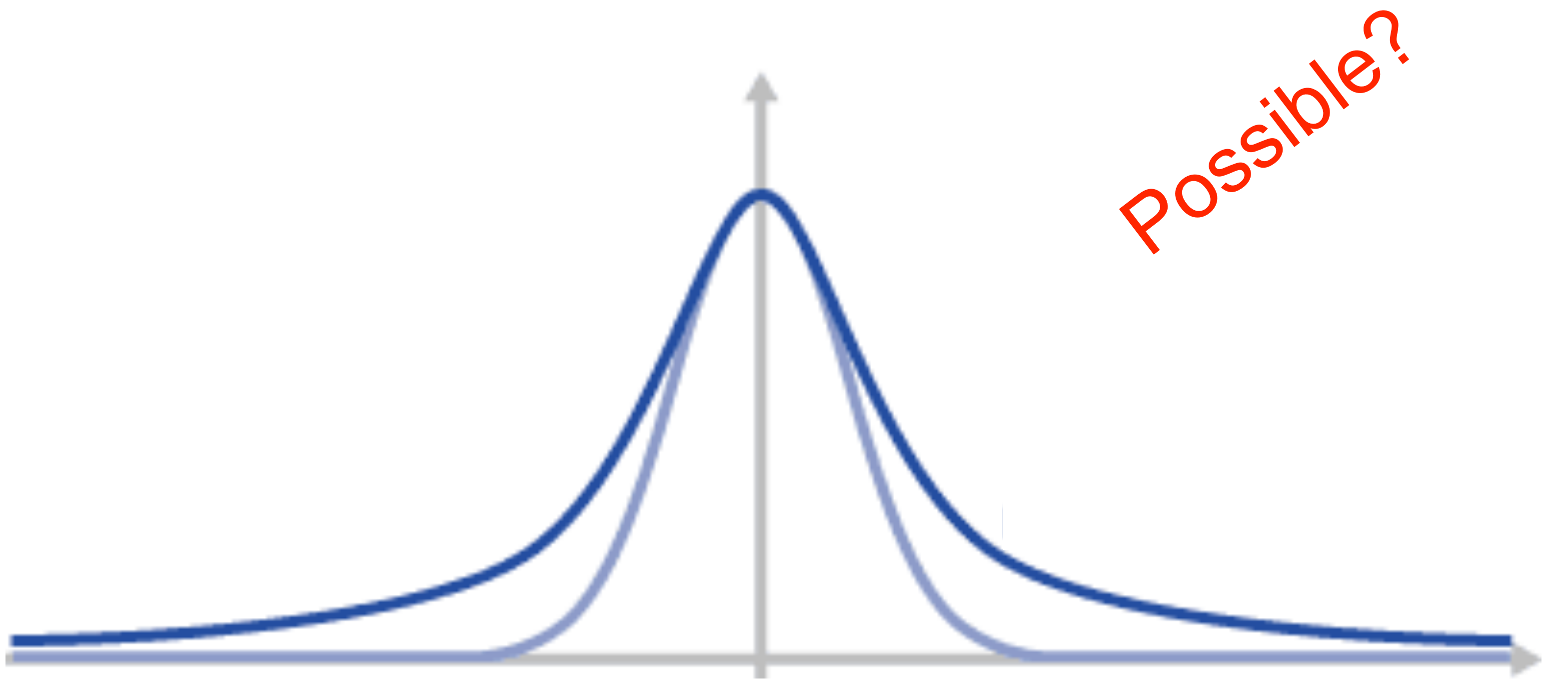
For a random variable X, the *tails* of X are the parts of the PMF that are “far” from its mean.



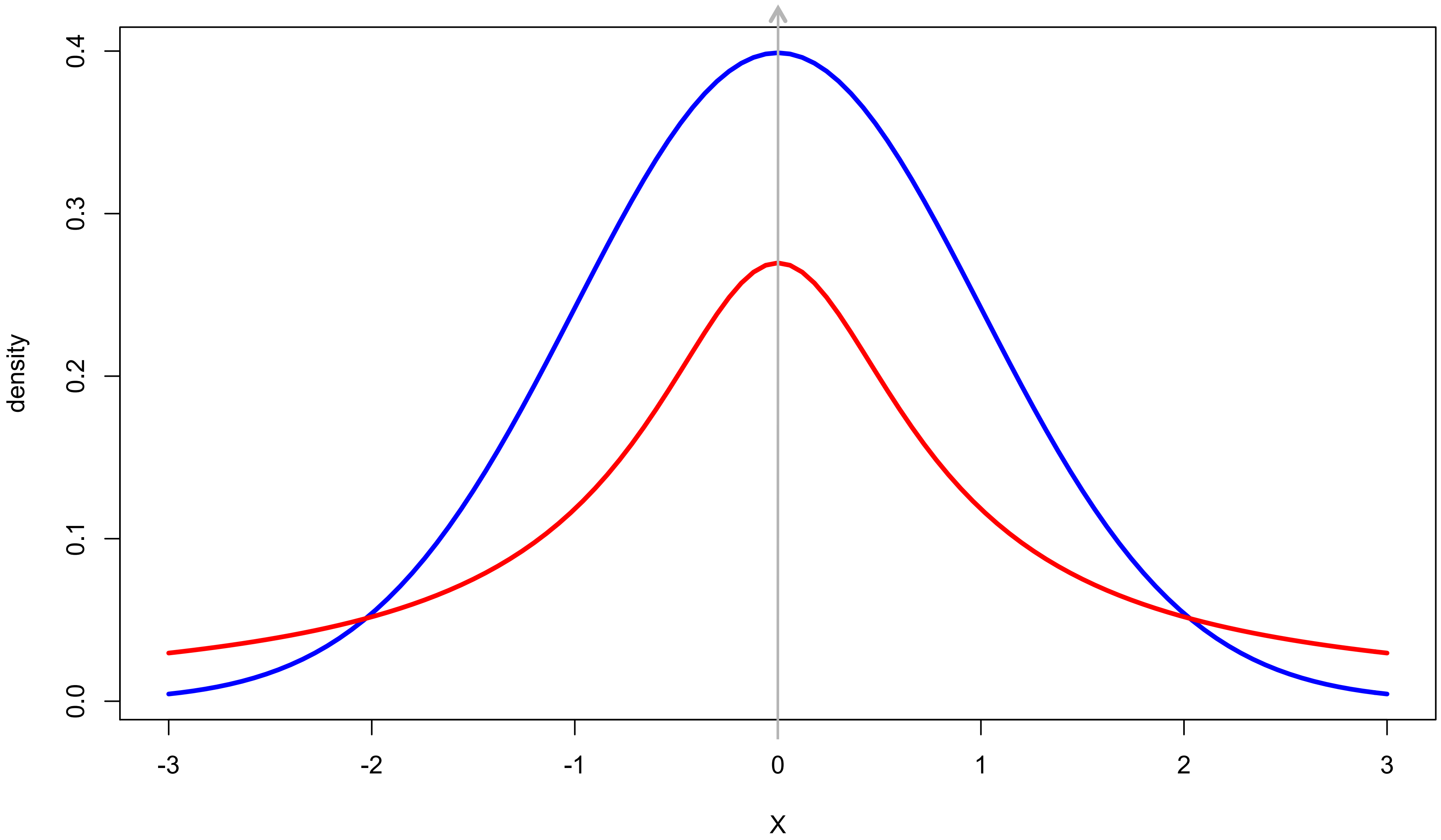
binomial tails



heavy-tailed distribution



heavy-tailed distribution



Often, we want to bound the probability that a random variable X is "extreme." Perhaps:

$$P(X > \alpha) < \frac{1}{\alpha^3}$$

$$P(X > E[X] + t) < e^{-t}$$

$$P(|X - E[X]| > t) < \frac{1}{\sqrt{t}}$$

We know that randomized quicksort runs in O(n log n) *expected* time. But what's the probability that it takes more than 10 n log(n) steps? More than $n^{1.5}$ steps?

If we know the expected advertising cost is $1500/day, what's the probability we go over budget? By a factor of 4?

I only expect 10,000 homeowners to default on their mortgages. What's the probability that 1,000,000 homeowners default?

"Lake Wobegon, Minnesota, where
all the women are strong,
all the men are good looking,
and
all the children are above average…"

In general, an *arbitrary* random variable could have very bad behavior. But knowledge is power; if we know *something*, can we bound the badness?

Suppose we know that X is always non-negative.

Theorem: If X is a non-negative random variable, then for every $\alpha > 0$, we have

$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

Corr:

$$P(X \geq \alpha E[X]) \leq 1/\alpha$$

Theorem: If X is a non-negative random variable, then for every $\alpha > 0$, we have

$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

Example: if X = daily advertising expenses and

E[X] = 1500

Then, by Markov's inequality,

$$P(X \geq 6000) \leq \frac{1500}{6000} = 0.25$$

Theorem: If X is a non-negative random variable, then for every $\alpha > 0$, we have

$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

Example: if X = time to quicksort *n* items, expectation E[X] ≈ 1.4 *n* log *n*. What's probability that it takes > 4 times as long as expected?

By Markov's inequality:

$$P(X \geq 4 \cdot E[X]) \leq E[X]/(4\,E[X]) = 1/4$$

Theorem: If X is a non-negative random variable, then for every $\alpha > 0$, we have

$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

Proof:

$$E[X] = \Sigma_x xP(x)$$

$$= \Sigma_{x<\alpha} xP(x) + \Sigma_{x\geq\alpha} xP(x)$$

$$\geq \quad 0 \quad + \Sigma_{x\geq\alpha} \alpha P(x)$$

$(x \geq 0;\ \alpha \leq x)$

$$= \alpha P(X \geq \alpha)$$

Theorem: If X is … tive random variable, then fo … have

Proof:

E[X] …

= …

≥ …)   (x ≥ 0; α ≤ x)

= αP(X …

If we know *more* about a random variable, we can often use that to get *better* tail bounds.

Suppose we *also* know the variance.

Theorem: If Y is an arbitrary random variable with E[Y] = μ, then, for any $\alpha > 0$,

$$P(|Y-\mu| \geq \alpha) \leq \frac{\mathrm{Var}[Y]}{\alpha^2}$$

Theorem: If Y is an arbitrary random variable with $\mu = E[Y]$, then, for any $\alpha > 0$,

$$P(|Y - \mu| \geq \alpha) \leq \frac{\mathrm{Var}[Y]}{\alpha^2}$$

Proof: $\mathrm{Let}\ X = (Y - \mu)^2$

X is non-negative, so we can apply Markov's inequality:

$$P(|Y - \mu| \geq \alpha) = P(X \geq \alpha^2)$$

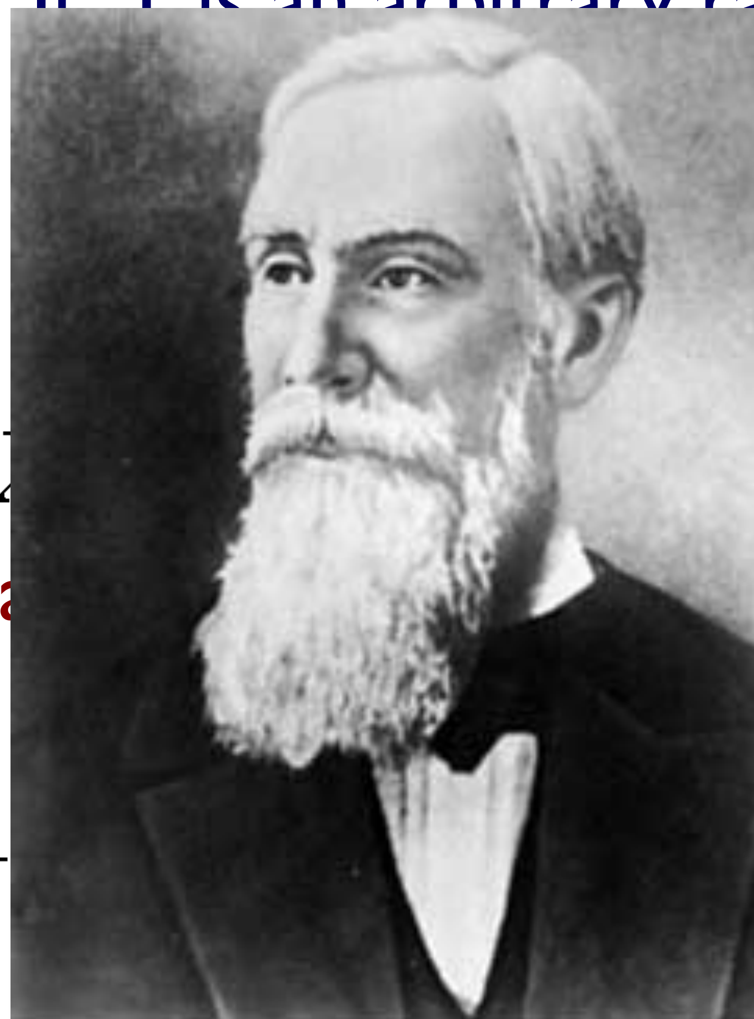
$$\leq \frac{E[X]}{\alpha^2} = \frac{\mathrm{Var}[Y]}{\alpha^2}$$

Theorem: If Y is an arbitrary random variable with … y $\alpha > 0$,

$$P(|Y \quad \cdots \quad \frac{\mathrm{Var}[Y]}{\alpha^2}$$

Proof: Let X …

X is non-neg… Markov's inequality:



$$P(|Y - \quad \cdots \quad \geq \alpha^2)$$

$$\leq \frac{\quad\cdots\quad}{\alpha^2} = \frac{\mathrm{Var}[Y]}{\alpha^2}$$

$$P(|Y - \mu| \geq \alpha) \leq \frac{\mathrm{Var}[Y]}{\alpha^2}$$

E.g., suppose:

Y = money spent on advertising in a day

E[Y] = 1500

Var[Y] = 500^2 (i.e. SD[Y] = 500)

$$P(Y \geq 6000) = P(|Y - \mu| \geq 4500)$$

$$\leq \frac{500^2}{4500^2} = \frac{1}{81} \approx 0.012$$

$$P(|Y-\mu| \geq \alpha) \leq \frac{\mathrm{Var}[Y]}{\alpha^2}$$

E.g., suppose:

Y = comparisons in quicksort for n=1024

$E[Y] = 1.4\ n \log_2 n \approx 14000$

$\mathrm{Var}[Y] = ((21-2\pi^2)/3)*n^2 \approx 441000$

(i.e. $SD[Y] \approx 664$)

$P(Y \geq 4\mu) = P(Y-\mu \geq 3\mu) \leq \mathrm{Var}(Y)/(9\mu^2) < .000242$

1000 times smaller than Markov

but still overestimated?: $\sigma/\mu \approx 5\%$, so $4\mu \approx \mu+60\sigma$

Theorem: If Y is an arbitrary random variable with $\mu = E[Y]$, then, for any $\alpha > 0$,

$$P(|Y - \mu| \geq \alpha) \leq \frac{\mathrm{Var}[Y]}{\alpha^2}$$

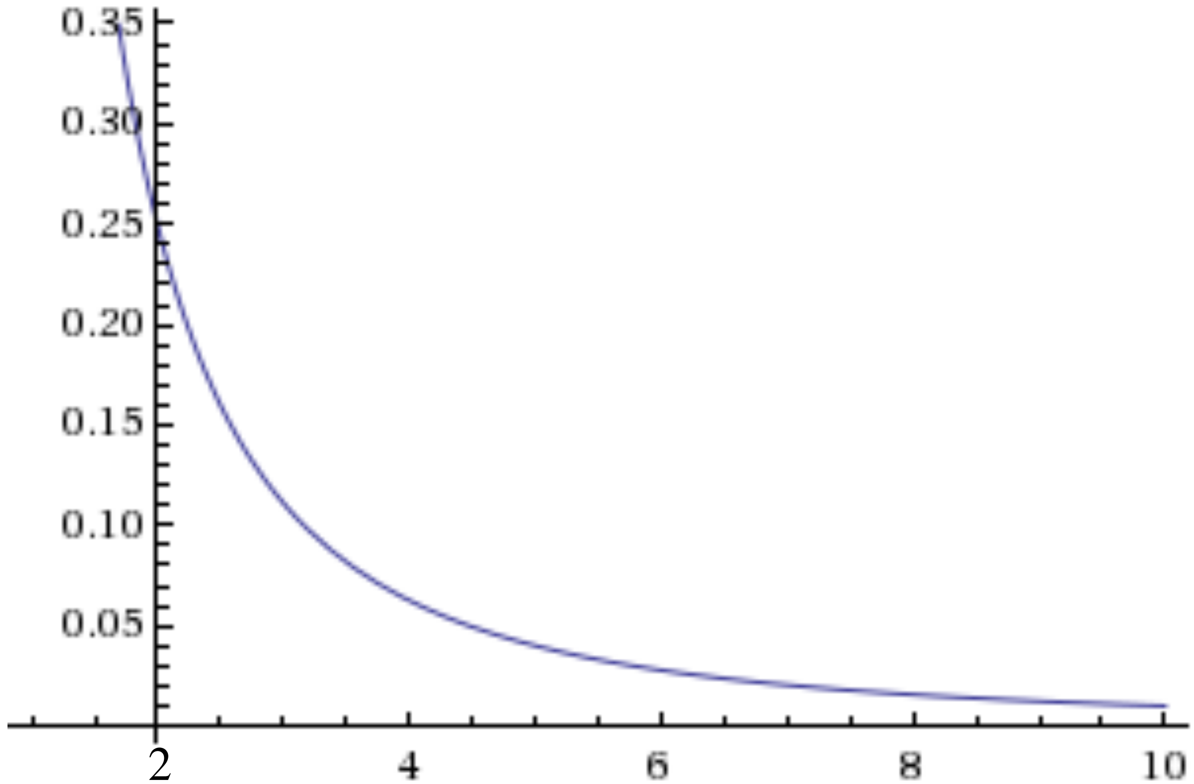
Corr: If

$$\sigma = SD[Y] = \sqrt{\mathrm{Var}[Y]}$$

Then:

$$P(|Y - \mu| \geq t\sigma) \leq \frac{\sigma^2}{t^2\sigma^2} = \frac{1}{t^2}$$

Chebyshev's inequality



$$P(|Y - \mu| \geq t\sigma) \leq \frac{1}{t^2}$$

For comparison, normal & many others would decline *exponentially* in t, or faster

I.e., Chebyshev is much weaker, but much more general

$Y \sim \mathrm{Bin}(15000, 0.1)$

$\mu = E[Y] = 1500, \sigma = \sqrt{\mathrm{Var}(Y)} = 36.7$

$P(Y \geq 6000) = P(Y \geq 4\mu) \leq 1/4$ (Markov)

$P(Y \geq 6000) = P(Y\text{-}\mu \geq 122\sigma) \leq 7\text{x}10^{-5}$ (Chebyshev)

Poisson approximation: $Y \sim \mathrm{Poi}(1500)$

Rough computer calculation:

$$P(Y \geq 6000) \ll 10^{-1600}$$

And the exact value is $\approx 4 \times 10^{-2031}$

Suppose X ~ Bin(n,p)
μ = E[X] = pn

Chernoff bound:

For any $0 < \delta < 1$,

$$P(X > (1+\delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{3}\right)$$

$$P(X < (1-\delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{2}\right)$$

Chernoff bounds

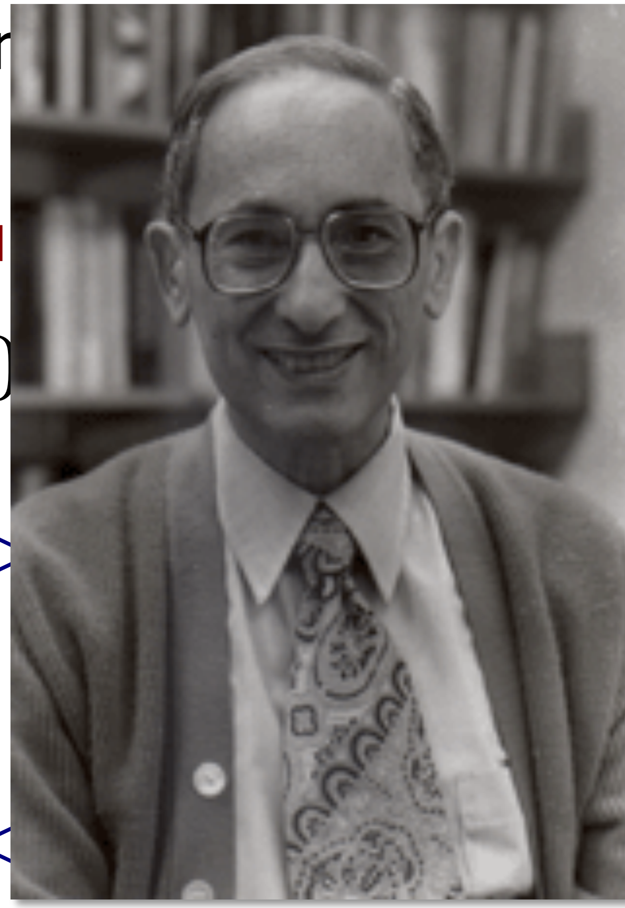
B&T pp 284-7

Suppose X ~ Bin(n,p)

μ = E[X] = pr

Chernoff bou

For any 0



$$P(X > \quad \text{xp}\left(-\frac{\delta^2\mu}{3}\right)$$

$$P(X < \quad \text{xp}\left(-\frac{\delta^2\mu}{2}\right)$$


COURTESY: CISCO

Model: n = 100,000 computers each independently send a packet with probability p = 0.01 each second. The router processes its buffer every second. How many packet buffers so that router drops a packet:

- Never?

  100,000
- With probability ≈1/2, every second?

  ≈1000 (P(X>E[X]) ≈ ½ when X ~ Binomial(100000, .01))
- With probability at most 10^{-6}, every hour?

  1257
- With probability at most 10^{-6}, every year?

  1305
- With probability at most 10^{-6}, since Big Bang?

  1404

Exercise: How would you formulate the exact answer to this problem in terms of binomial probabilities? Can you get a numerical answer?

X ~ Bin(100,000, 0.01), μ = E[X] = 1000

Let p = probability of buffer overflow in 1 second

By the Chernoff bound

$$p = P(X > (1+\delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{3}\right)$$

Overflow probability in n seconds

$= 1-(1-p)^n \leq np \leq n \exp(-\delta^2\mu/3)$,

which is $\leq \varepsilon$ provided $\delta \geq \sqrt{(3/\mu)\ln(n/\varepsilon)}$.

For $\varepsilon = 10^{-6}$ per hour: $\delta \approx .257$, buffers = 1257

For $\varepsilon = 10^{-6}$ per year: $\delta \approx .305$, buffers = 1305

For $\varepsilon = 10^{-6}$ per 15BY: $\delta \approx .404$, buffers = 1404

Tail bounds – bound probabilities of extreme events

Important, e.g., for "risk management" applications

Three (of many):

- Markov: $P(X \geq k\mu) \leq 1/k$ (weak, but general; only need $X \geq 0$ and μ)
- Chebyshev: $P(|X-\mu| \geq k\sigma) \leq 1/k^2$ (often stronger, but also need σ)
- Chernoff: various forms, depending on underlying distribution; usually 1/exponential, *vs* 1/polynomial above
- Generally, more assumptions/knowledge $\Rightarrow$ better bounds

"Better" than exact distribution?

- Maybe, e.g. if latter is unknown or mathematically messy

"Better" than, e.g., "Poisson approx to Binomial"?

- Maybe, e.g. if you need rigorously "≤" rather than just "≈"