A Demo or Two

May 18, 2015

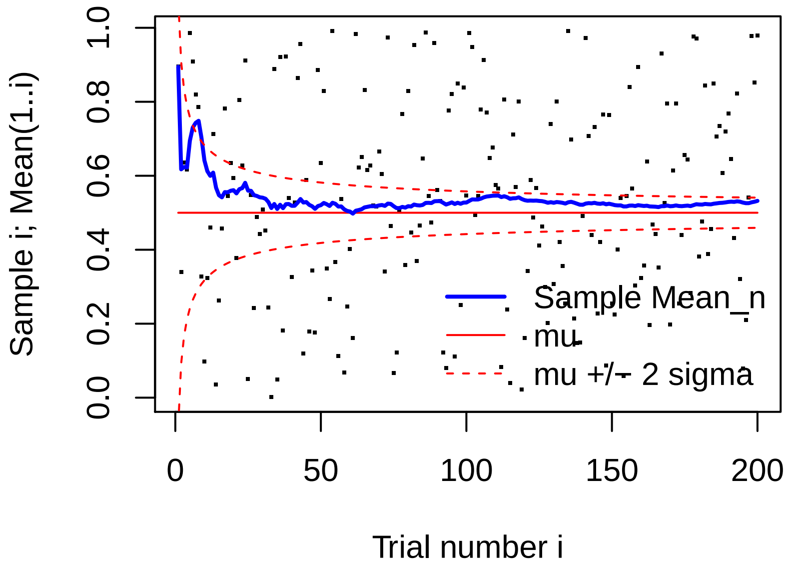
Here are a few simple demonstrations illustrating important concepts from the course. Most use R; see R Quick Start for a quick introduction to R. You should be able to run these demos below by copying the R code shown below and pasting it into an R console window.

Laws Of Large Numbers: The Weak and Strong Laws of Large Numbers are important theoretical results, essentially guaranteeing that the average of a large number if independent samples from arbitrary distributions will converge to the expected value of such a variable. As a simple illustration of this, the following looks at averages of i.i.d. Uniform(0,1) random variables:

```
# Helper Function: Running sum of a vector
runsum <- function(x){
  t <- x
  for(i in 2:length(t)){
    t[i]<- t[i-1]+t[i]
  }
  return(t)
}
```

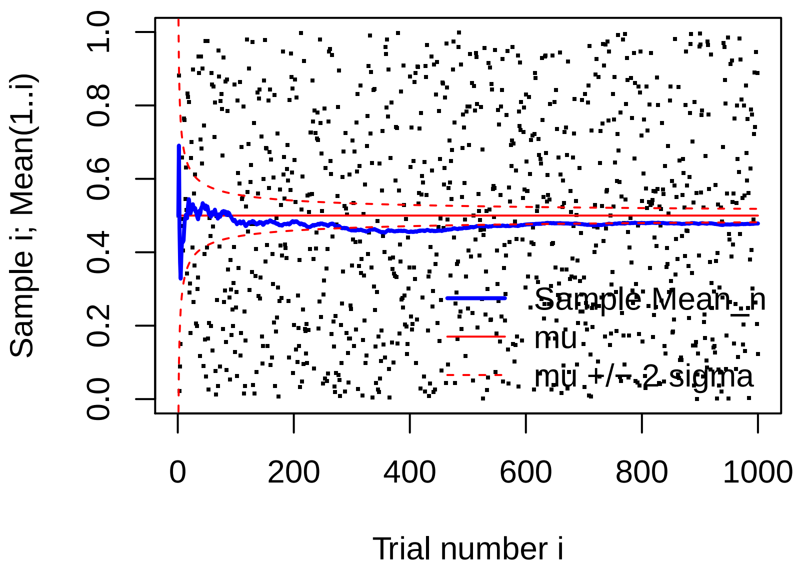
```
# "Regression Towards the Mean"
#
# Plot the mean of an increasingly large sample of uniform RVs;
# it should converge to the mean
#
# Parameters:
#   n = # samples,
#   cex: controls point size,
#   ksigma: if ksigma > 0, also plot +/- k*sigma envelope around the mean
#
rtm <- function(n=200,ksigma=2,cex=2,cmu='red',cavg='blue',csig='red'){
  v <- runif(n)
  mu <- 0.5
  sigma <- 1/sqrt(12)
  plot(v,pch='.',xlab='Trial number i',ylab='Sample i; Mean(1..i)',cex=cex)
  lines(c(1,n),c(mu,mu),col=cmu,lwd=1)
  points(1:n,runsum(v)/(1:n),type='l',col=cavg,lwd=2)
  if(ksigma>0){
    points(1:n, mu+ksigma*sigma/sqrt(1:n), type='l',lwd=1,col=csig,lty='dashed')
    points(1:n, mu-ksigma*sigma/sqrt(1:n), type='l',lwd=1,col=csig,lty='dashed')
    legend('bottomright', bty='n',
           legend = c("Sample Mean_n",    "mu", paste("mu +/-",ksigma,"sigma")),
           col    = c(           cavg,     cmu,                           csig),
           lwd    = c(              2,       1,                              1),
           lty    = c(        'solid','solid',                        'dashed')
          )
  }
}
```

```r
rtm()
```



Another plot, with larger n:

```r
rtm(n=1000)
```



Exercise: Do something similar for a different distribution (normal, exponential, Poisson,...) in place of uniform.

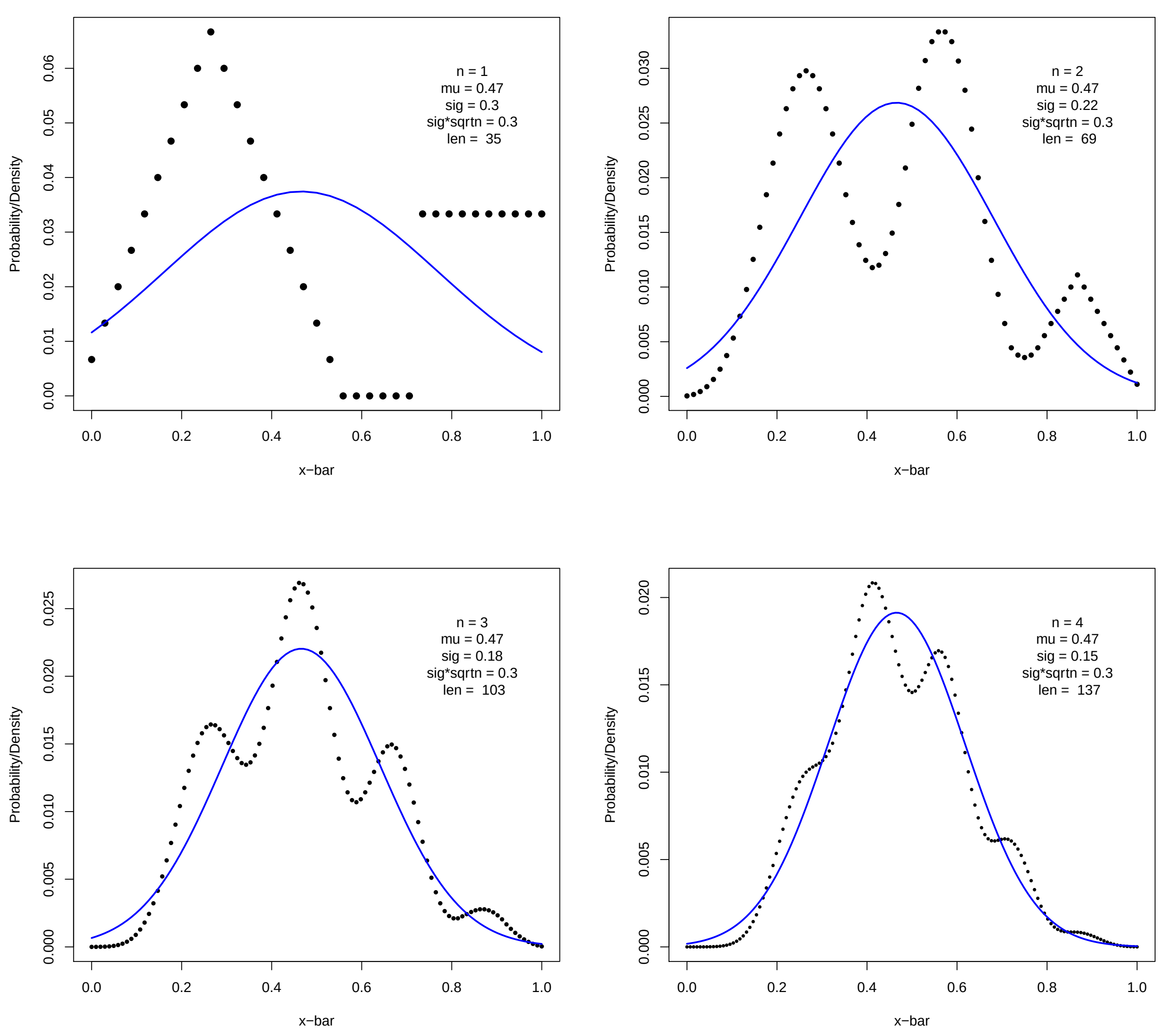
The Central Limit Theorem: Another very important result is the Central Limit Theorem: Not only does the *value* of the average of a large sample of independent random variables from *arbitrary* distributions converge to its expected limit (above), but *the shape of the distribution* of those averages also converges to a well-defined limit—namely, it is approximately normally distributed.

```
# Convergence of any whacky distrib to normal as in CLT
#
# Method: n-fold convolution of initial distribution with itself
#
#   For the n-th convolution, we need both the (n-1)-st and the original
#   distributions, so for convenience these are bundled into a list and
#   returned, making a by-hand iteration simple:
#     bundle1 <- clt(whacky=//*put your whacky distribution here*//)
#     bundle2 <- clt(bundle1) # 2-fold convolution
#     bundle3 <- clt(bundle2) # 3-fold convolution
#     ...
#
# Parameters:
#    bundle  : initially NULL; subsequently, result of previous call
#    plot    = T to see plot
#    verbose = T to annotate plot with mu, sigma, etc.
#    bell    = T to overlay bell curve
#    whacky  = vector of numbers representing relative probabilities of
#              outcomes 1:length(whacky); irrelevant unless bundle == NULL
#    cex     = scale factor for point size
#
clt <- function(bundle=NULL, plot=T, verbose=T, bell=T,
                whacky=c(1:10,9:0,rep(0,5),rep(5,10)), cex=NULL){
  if(is.null(bundle)){
    mywhack <- whacky/sum(whacky)                    # normalize
    bundle <- list(n=1,result=mywhack,start=mywhack) # bundle params/result
  }
  if(plot){
    len <- length(bundle$result)
    x <- (0:(len-1))/(len-1)
    y <- bundle$result
    plot(x,y,xlab='x-bar',ylab='Probability/Density',cex=cex,pch=19)
    mu   <- sum(x*y)
    sig2 <- sum((x-mu)^2*y)
    sig  <- sqrt(sig2)
    chatter <- ifelse(!verbose,'', paste(
      '\nmu =', round(mu,2),
      '\nsig =',round(sig,2),
      '\nsig*sqrtn =', round(sig*sqrt(bundle$n),2),
      '\nlen = ', length(x)));
    text(.85,.8*max(bundle$result), paste('n =',bundle$n,chatter))
    if(bell){points(x,dnorm(x,mu,sig)/length(x),type='l',lwd=2,col='blue')}
  }
  return(
    list(
      n      = bundle$n+1,
      result = convolve(bundle$result,rev(bundle$start),type='o'),
      start  = bundle$start))
}
```

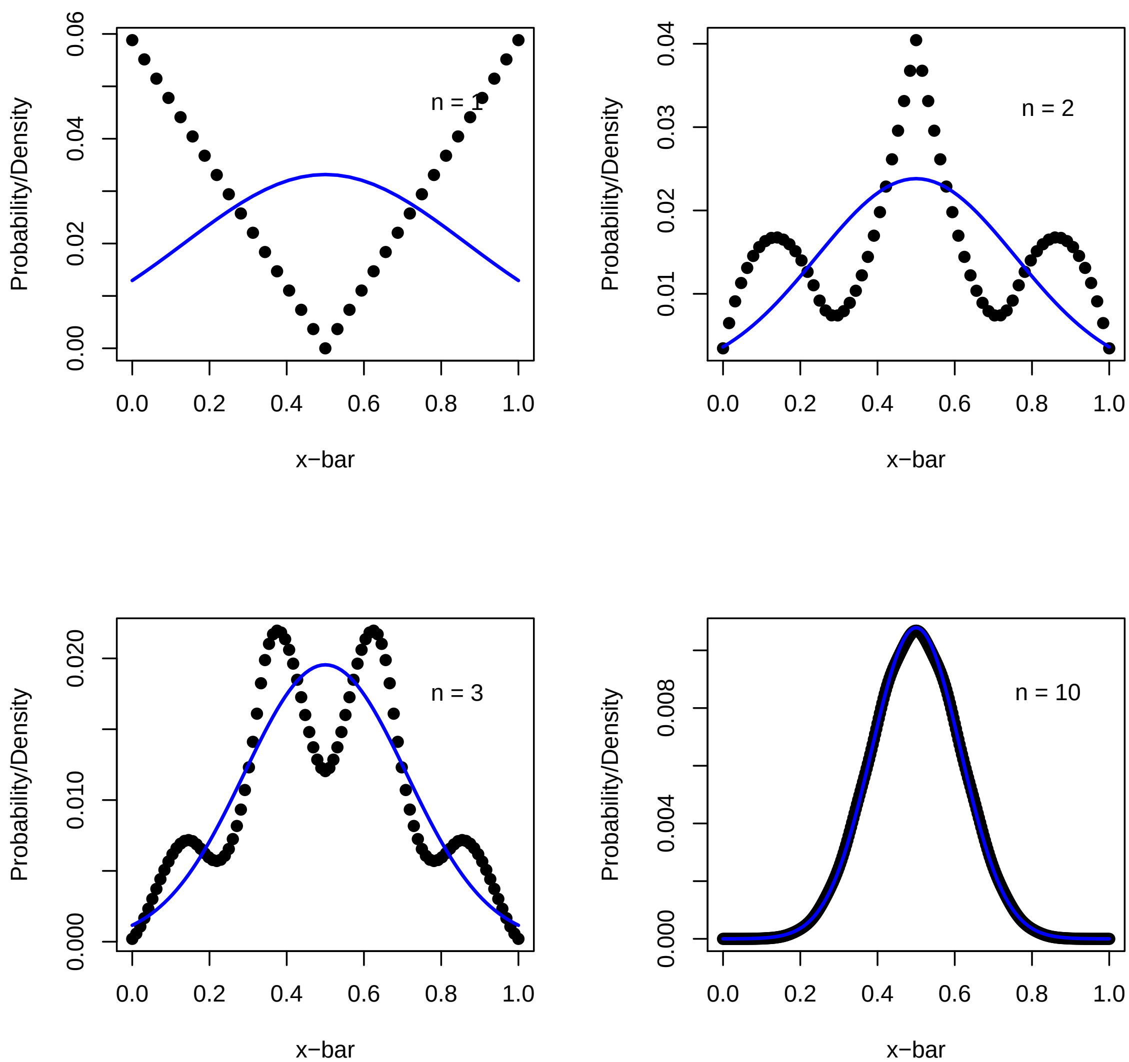
```
# Make a "movie" of above; if "file" is NULL, display to screen, else write a
# multi-page .pdf file. The "..." formal and actual parameters have a special
# meaning in R: accept extra named arguments to this function and pass them to
```

```
# inner calls.
clt.movie <- function(filename ="central.limit.thm.movie.pdf", frames=50, ...){
  opar<-par(no.readonly=T);on.exit(par(opar))
  if(!is.null(filename)){
    # noninteractive version: open .pdf graphics "device"
    pdf(filename,onefile=T,width=9,height=7)
  } else {
    # interactive version: pause after each plot & ask to continue
    devAskNewPage(TRUE)
  }
  bundle <- clt(...)
  for(i in 2:frames){
    # tweak cex to make dots smaller when there are more of them
    bundle <- clt(bundle, cex=(1-i/frames)*.6+.4, ...)
  }
  if(!is.null(filename)){dev.off()}  # close .pdf
}
```

```
clt.movie(NULL,4)
```



```r
# Another CLT example, just showing 4 of 10 frames
# Default is a vee-shaped distribution
clt.vee <- function(dist=abs(-16:16), ...){
  opar <- par(mfrow=c(2,2),no.readonly=T) # graph params: 4 plots in 2x2 grid
  on.exit(par(opar))
  bundle <- clt(whacky=dist, plot=TRUE, ...)
  for(i in 2:10) {
    # show plots only for 1-, 2-, 3-, and 10-fold convolution
    bundle <- clt(bundle, plot = (i %in% c(2,3,10)), ...)
  }
}
clt.vee(verbose=F)
```



Exercises: The above should work for any disctrete distribution defined on a finite number of points. Try it on some other ones. Try to find ones that make the convergence to the normal as slow as possible, say.