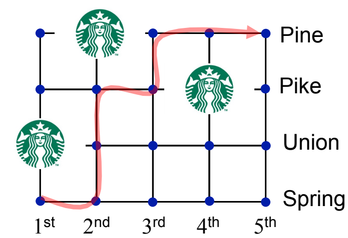
Counting

counting is hard with only 10 fingers

How many ways to do **X**?

X = "Choose an integer between one and ten."

X = "Walk from 1st and Spring to 5th and Pine."



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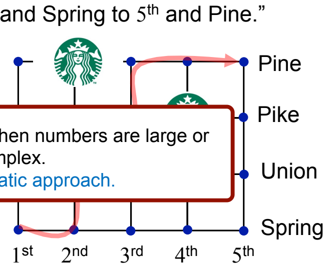
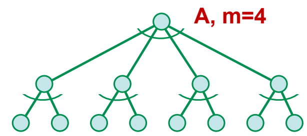


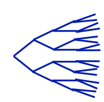
Counting is hard when numbers are large or constraints are complex.
We need a systematic approach.

the basic principle of counting (product rule)

If there are **m** outcomes from some event **A**, followed sequentially by **n** outcomes from some event **B**, then there are… **m x n** outcomes overall.



Generalizes to more events.



examples

How many n-bit numbers are there?

$2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$

How many subsets of a set of size n are there?

{1, 2, 3, …, n}

Set contains 1 or doesn't contain 1.
Set contains 2 or doesn't contain 2.
Set contains 3 or doesn't contain 3...

$2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$

examples

How many 4-character passwords are there if each character must be one of
a, b, c, …, z, 0, 1, 2, …, 9 ?

$36 \cdot 36 \cdot 36 \cdot 36 = 1{,}679{,}616 \approx 1.7$ million

Same question, but now characters cannot be repeated…

$36 \cdot 35 \cdot 34 \cdot 33 = 1{,}413{,}720 \approx 1.4$ million

permutations

How many arrangements of the letters {a,b,c} are possible
(using each once, no repeat, order matters)?

a b c	b a c	c a b
a c b	b c a	c b a

More generally, how many arrangements of n distinct items are possible?

n • (n-1) • (n-1) • ... • 1 = n!   (n factorial)

permutations

Q. How many permutations of DOGIE are there?

5! = 120

Q. How many of DOGGY ?

5!/2! = 60

DOG_1G_2Y
DOG_2G_1Y

Q. How many of GODOGGY ?

$$\frac{7!}{3!2!1!1!} = 420$$

combinations

Your dark elf avatar can carry three objects chosen from:

How many ways can he/she be equipped?

$$\frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3! \cdot 2!} = 10$$

combinations

Combinations: Number of ways to choose r things from n things

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Pronounced "n choose r"   aka   "binomial coefficients"

E.g., $\binom{n}{2} = \frac{n(n-1)}{2} = \Theta(n^2)$

Many identities:

$\binom{n}{r} = \binom{n}{n-r}$ ← by symmetry of definition

$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ ← 1st object either in or out

$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$ ← team + captain

the binomial theorem

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$

Proof 1: Induction …

Proof 2: Counting

(x+y) • (x+y) • (x+y) • ... • (x+y)

Pick either x or y from first factor
Pick either x or y from second factor
…
Pick either x or y from nth factor

How many ways to get exactly k x's? $\binom{n}{k}$

an identity with binomial coefficients

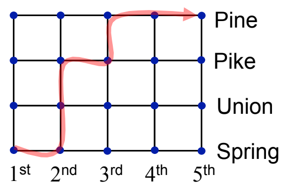
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

counting paths

How many ways to walk from 1st and Spring to 5th and Pine only going North and East?

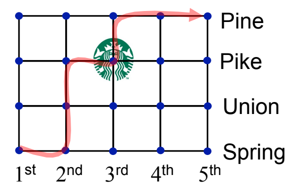


A: *Changing the visualization often helps.*
Instead of tracing paths on the grid above, list choices. You walk 7 blocks; at each intersection choose N or E; must choose N exactly 3 times.

$$\binom{7}{3} = 35$$

counting paths

How many ways to walk from 1st and Spring to 5th and Pine only going North and East, if I want to stop at Starbucks on the way?



Other problems

10 people of different heights. How many ways to line up 5 of them?

Line up 5 of them in height order?

# of ways to rearrange letters in word SYSTEMS

Other problems

of 7 digit numbers (decimal) with at least one repeating digit? (allowed to have leading zeros).