Probability theory: way to mathematically model uncertain situations

$\boxed{\text{Sample space } \Omega:}$ set of possible outcomes of random experiment

Example: result of tossing coin 3 times

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

possible outcome

$|\Omega| = 8$

$\boxed{\text{Probability associated to each } \omega \in \Omega}$

1. $0 \leq \Pr(\omega) \leq 1 \quad \forall \omega \in \Omega$

2. $\sum_{\omega \in \Omega} \Pr(\omega) = 1$

Simplest way to assign probs: $\boxed{\text{Uniform}}$ each outcome equally likely

i.e. $\Pr(\omega) = \frac{1}{|\Omega|} \quad \forall \omega \in \Omega$

for 3 coin tosses $\Pr(\omega) = \frac{1}{8}$

Another example:

$\Omega = \{5 \text{ card poker hands}\}$ $\quad |\Omega| = \binom{52}{5}$

Key concept: event $E \subseteq \Omega$ subset of sample space

Examples: Sum of rolls of 2 dice ≥ 10

poker hand is flush

in 100 coin tosses, ≥ 33 heads

$\forall E \subseteq \Omega \qquad Pr(E) = \sum_{\omega \in E} Pr(\omega)$

$Pr(2\ H\text{'s in } 3 \text{ coin tosses}) = Pr(\{HHT,\ HTH,\ THH\}) = \frac{3}{8}$

If E & F mutually exclusive events

$$Pr(E \cup F) = Pr(E) + Pr(F)$$

Implications:

- $Pr(\bar{E}) = 1 - Pr(E)$
- If $E \subseteq F$ $\quad Pr(E) \leq Pr(F)$
- $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$

Probabity Space: Ω $\qquad$ P

flip fair coin $\quad \Omega = \{H, T\}$ $\quad Pr(H) = Pr(T) = \frac{1}{2}$

flip fair coin 100 times $\quad \Omega = \{$ all possible sequences of heads & tails of length 100 $\}$ $\quad Pr(\omega) = \frac{1}{2^{100}}$

$Pr(\text{all 100 tosses same}) = \frac{1}{2^{100}} + \frac{1}{2^{100}} = \frac{1}{2^{99}}$

flip biased coin 100 times bias $= \frac{2}{3}$ $\quad \Omega = \{$ all possible sequences of heads & tails of length 100 $\}$ $\quad Pr(\omega) = ?$

roll 2 fair dice $\quad \Omega = \{(i,j) : 1 \le i \le 6, 1 \le j \le 6\}$ $\quad Pr((i,j)) = \frac{1}{36}$

$|\Omega| = 36$

$Pr(i+j=5) = Pr(\{(1,4),(2,3),(3,2),(4,1)\}) = \frac{4}{36} = \frac{1}{9}$

(Ω, P) uniform prob space

$$Pr(E) = \frac{|E|}{|\Omega|} \quad \forall \text{ event } E$$

Shuffle deck of 52 cards

Ω = all permutations of deck

$|\Omega| = 52!$

$Pr(\text{particular perm}) = \frac{1}{52!}$

5 card Poker hands

Ω = all possible 5 card hands

$|\Omega| = \binom{52}{5}$

$Pr(\text{particular hand}) = \frac{1}{\binom{52}{5}}$

$$\underset{\substack{\uparrow \\ \text{all cards have same suit}}}{Pr(\text{flush})} = \frac{|F|}{|\Omega|} = \frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} \approx 0.02$$

20 labeled balls $\rightarrow$ 10 labeled bins

$$\Omega = \{(b_1, b_2, \ldots, b_{20}) \mid 1 \le \underset{\substack{\uparrow \\ \text{which bin ball } i \text{ goes to}}}{b_i} \le 10\}$$

$Pr(\omega) = \frac{1}{10^{20}}$

$|\Omega| = 10^{20}$

$$Pr(\overbrace{\text{bin 1 is empty}}^{E}) = \frac{|\text{outcomes with bin 1 empty}|}{|\Omega|} = \frac{9^{20}}{10^{20}} \approx 0.12$$

$$Pr(\underbrace{\text{bin 1 contains at least one ball}}_{\overline{E}}) = 1 - Pr(E)$$

≈ 0.88

Key steps in prob calculations:

1. What is sample space (experiment & set of possible outcomes)?

2. What is probability of each outcome (sample pt)?

3. What event are we interested in? (which subset of sample space?)

4. Compute probability of event by adding up probs of sample pts in it.