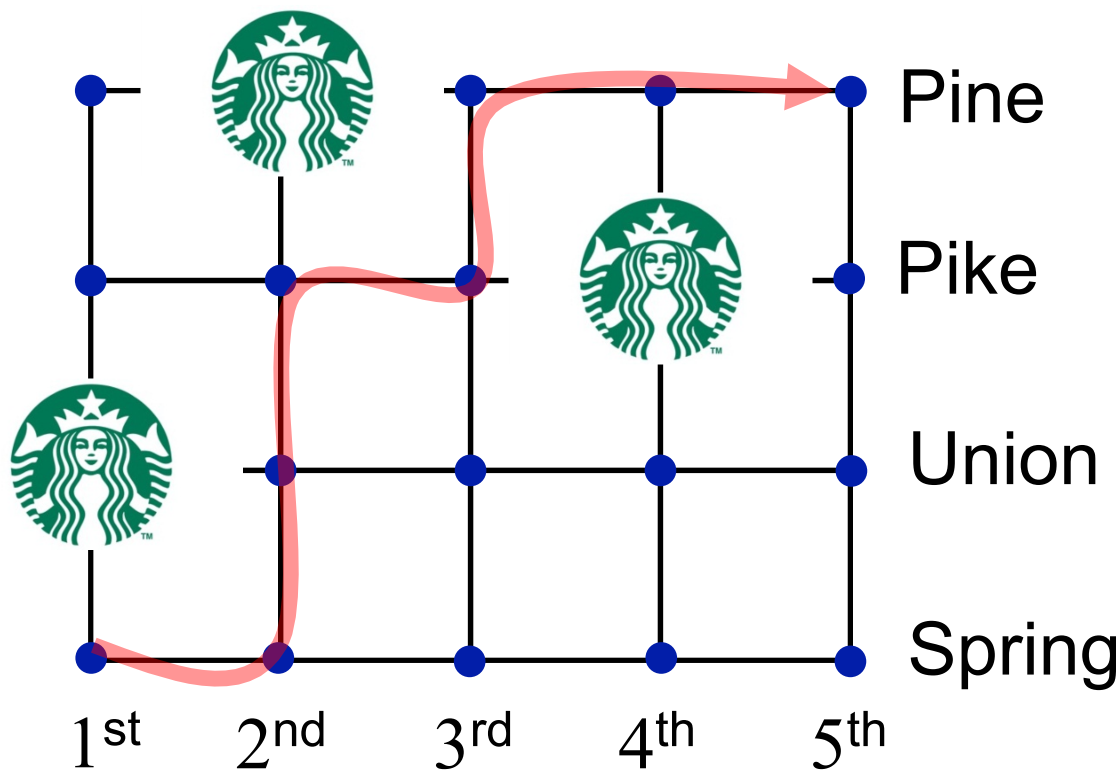
# Counting

How many ways to do **X**?

**X** = "Choose an integer between one and ten."

**X** = "Walk from 1st and Spring to 5th and Pine."



## counting is hard with only 10 fingers

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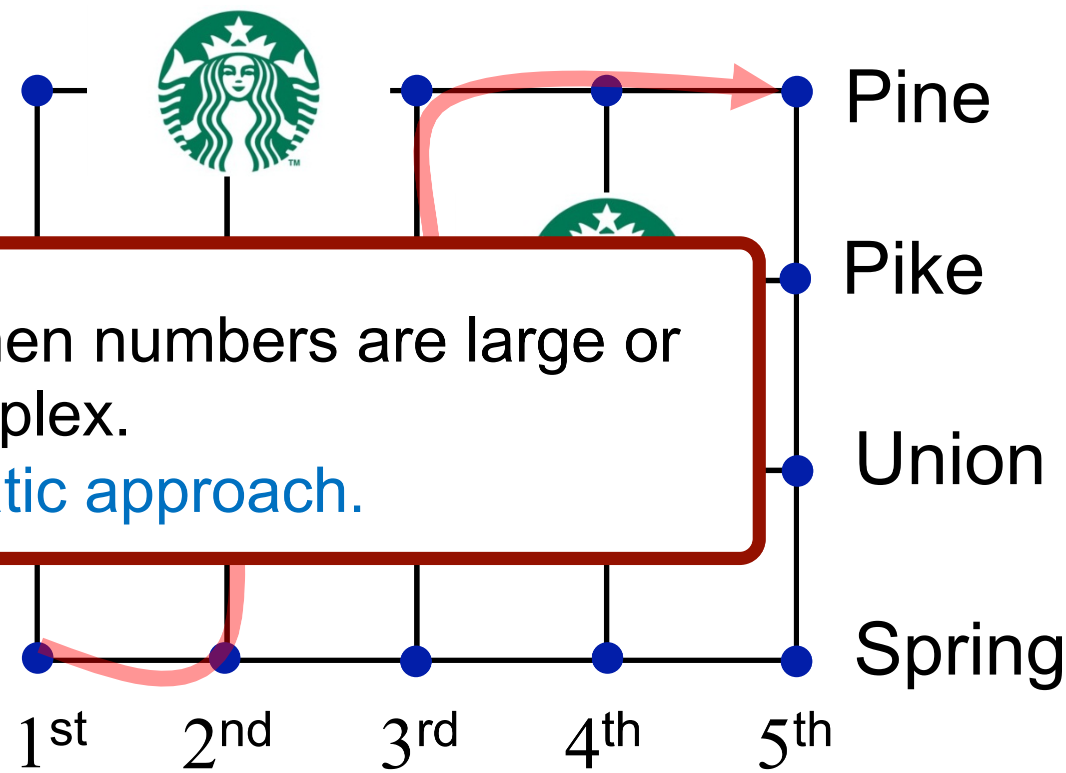
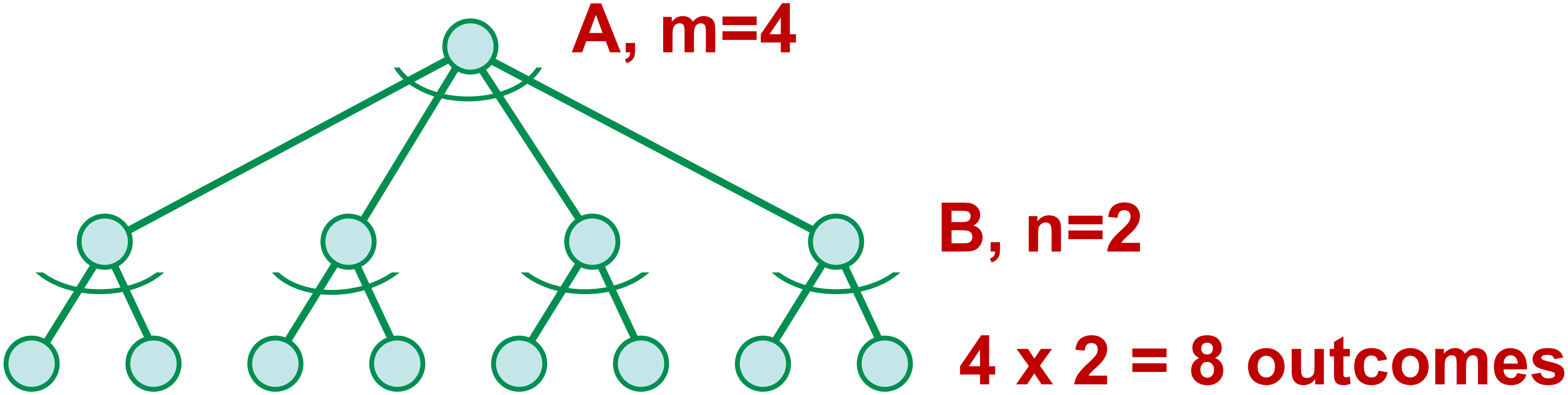



Counting is hard when numbers are large or constraints are complex.
We need a systematic approach.

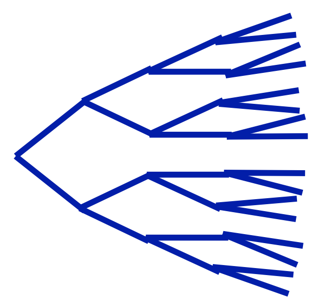
## the basic principle of counting (product rule)

If one of **m** outcomes is chosen from **A**, followed sequentially by one of **n** outcomes from **B**, then there are…

**m x n** outcomes possible overall.




Generalizes to more events.



How many n-bit numbers are there?

$$2 \cdot 2 \cdot \ \ldots \ \cdot 2 = 2^n$$

How many subsets of a set of size n are there?

$\{1, 2, 3, \ldots, n\}$

Set contains **1** or doesn't contain **1**.

Set contains **2** or doesn't contain **2**.

Set contains **3** or doesn't contain **3**...

$$2 \cdot 2 \cdot \ \ldots \ \cdot 2 = 2^n$$

How many 4-character passwords are there if each character must be one of
a, b, c, …, z, 0, 1, 2, …, 9 ?

$36 \cdot 36 \cdot 36 \cdot 36 = 1{,}679{,}616 \approx 1.7$ million

Same question, but now characters cannot be repeated…

$36 \cdot 35 \cdot 34 \cdot 33 = 1{,}413{,}720 \approx 1.4$ million

How many arrangements of the letters {a,b,c} are possible
(using each once, no repeat, order matters)?

| | | |
|---|---|---|
| a b c | b a c | c a b |
| a c b | b c a | c b a |

More generally, how many arrangements of n distinct items are possible?

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n! \qquad \text{(n factorial)}$$

Q. How many permutations of DOGIE are there?

**5! = 120**

Q. How many of DOGGY ?

**5!/2! = 60**

$DOG_1G_2Y$
$DOG_2G_1Y$

Q. How many of GODOGGY ?

$$\frac{7!}{3!2!1!1!} = 420$$

# combinations

Your dark elf avatar can carry three objects chosen from:

How many ways can he/she be equipped?

$$\frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3! \cdot 2!} = 10$$

**Combinations:** Number of ways to choose r things from n things

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Pronounced "n choose r" aka "binomial coefficients"

E.g., $\binom{n}{2} = \frac{n(n-1)}{2} = \Theta(n^2)$

**Many identities:**

$$\binom{n}{r} = \binom{n}{n-r}$$ ← by symmetry of definition

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$ ← 1st object either in or out

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$ ← team + captain

$$\boxed{(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}}$$

**Proof 1:** Induction …

**Proof 2:** Counting

(x+y) • (x+y) • (x+y) • ... • (x+y)

Pick either x or y from first factor
Pick either x or y from second factor
…
Pick either x or y from nth factor

How many ways to get exactly k x's?

$$\binom{n}{k}$$

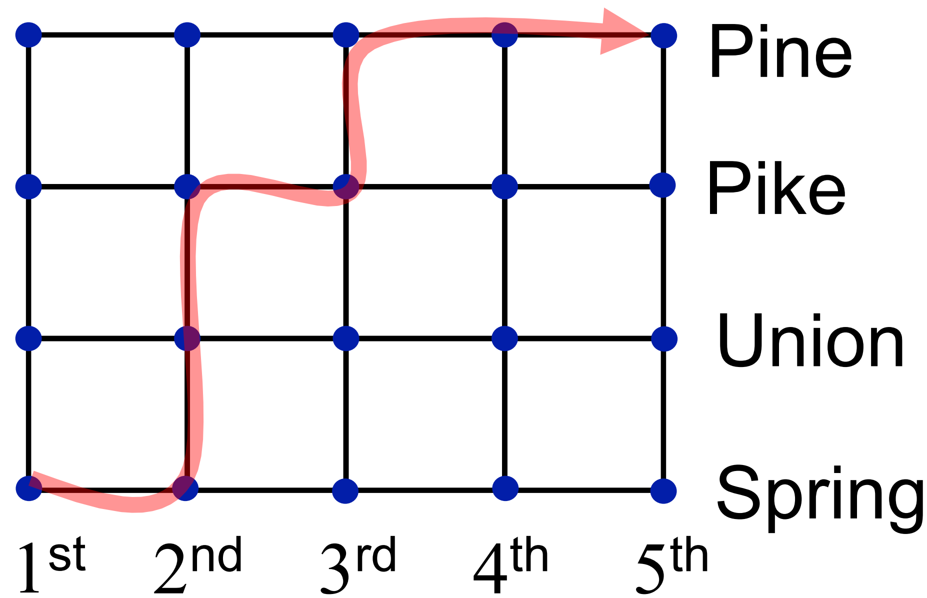
## an identity with binomial coefficients

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

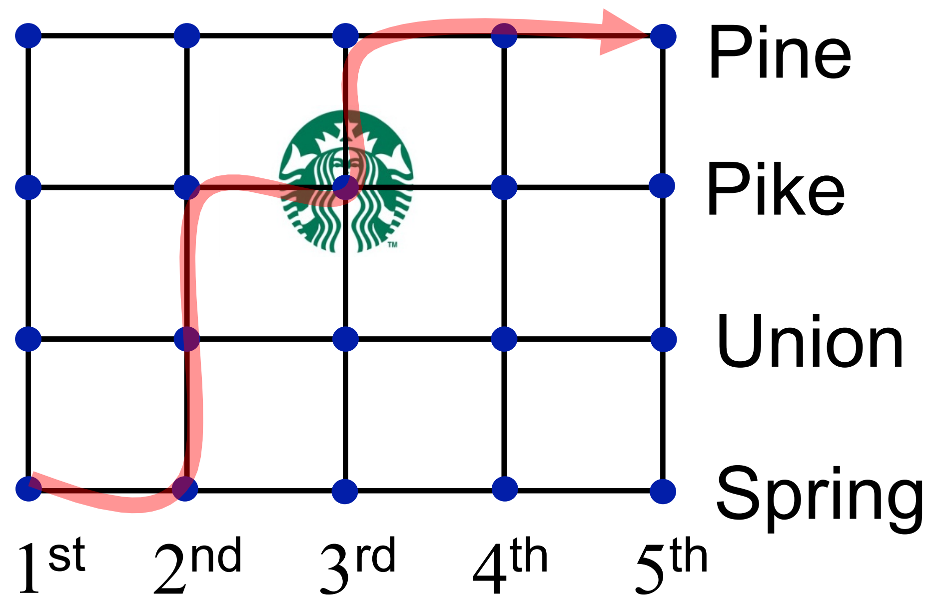
How many ways to walk from $1^{st}$ and Spring to $5^{th}$ and Pine only going North and East?




A: *Changing the visualization often helps.*
Instead of tracing paths on the grid above, list choices. You walk 7 blocks; at each intersection choose N or E; must choose N exactly 3 times.

$$\binom{7}{3} = 35$$

How many ways to walk from $1^{st}$ and Spring to $5^{th}$ and Pine only going North and East, if I want to stop at Starbucks on the way?



## Other problems

10 people of different heights. How many ways to line up 5 of them?

Line up 5 of them in height order?

# of ways to rearrange letters in word SYSTEMS

# Other problems

# of 7 digit numbers (decimal) with at least one repeating digit? (allowed to have leading zeros).