DIRECTIONS:

- Closed book, closed notes except for 1 handwritten 8.5×11 sheet.
- Time limit 110 minutes.
- Calculators allowed.
- No need to simplify your answers unless explicitly requested to. Grading will emphasize problem set-up over calculation.
- If possible, answer all problems on these sheets.
- Put your NAME on each sheet.
- Do not turn the page until I tell you to.
- Good luck!

<table>
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<tr>
<td></td>
<td>/34</td>
<td>/27</td>
<td>/28</td>
<td>/30</td>
<td>/50</td>
<td>/30</td>
<td>/25</td>
<td>/26</td>
<td>/250</td>
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The following short table of values of the cumulative distribution function for the standard normal distribution may be useful.

<table>
<thead>
<tr>
<th>x</th>
<th>-10</th>
<th>-8</th>
<th>-6</th>
<th>-4</th>
<th>-2</th>
<th>-1.96</th>
<th>-1.64</th>
<th>-1.00</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>ϕ(x) = P(X &lt; x)</td>
<td>7.62e-24</td>
<td>6.22e-16</td>
<td>9.87e-10</td>
<td>3.17e-5</td>
<td>0.0228</td>
<td>0.025</td>
<td>0.05</td>
<td>0.159</td>
<td>.5</td>
</tr>
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</table>
1. (34 points) Circle True or False. (No justification required.)

(a) For any two events $A$ and $B$, $P(A \cup B) \leq P(A) + P(B)$. .............................................. T F

(b) For any continuous probability distribution on the reals, $P(X = c) = 0$ for all values of $c \in \mathbb{R}$. T F

(c) If you toss a coin independently 100 times, with probability $3/4$ of coming up heads each time, you will always observe exactly 75 heads. .......................................................... T F

(d) If you toss a coin independently 100 times, with probability $3/4$ of coming up heads each time, the probability that you see 10 heads is $(3/4)^{10}(1/4)^{90}$ .......................................................... T F

(e) If you toss a coin independently 100 times, with probability $3/4$ of coming up heads each time, the probability that there are more than 85 heads is the same as the probability that there are less than 65 heads. .......................................................... T F

(f) For all random variables $X$, it is true that $E[X^2] \geq (E[X])^2$. .............................................. T F

(g) For all random variables $X$, it is true that $\text{Var}(X) > 0$. .............................................. T F

(h) A fair coin is tossed 500 times. If the first 100 tosses all land heads, the Law of Large Numbers implies that we expect to see 150 heads in the final 400 tosses. .............................................. T F

(i) If $Z$ is normally distributed with mean 0 and variance 4, and for some point $a$ on the number-line, $P(Z < -a) = P(Z > a)$. .......................................................... T F

(j) For any events $E$ and $F$ such that $P(E \cap F) > 0$, it holds that $P_r(E|E \cap F) \leq P_r(E|F)$. .............................................. T F

(k) Chebychev’s Inequality states that the proportion of observations that are within 2 standard deviations of the mean is at least 3/4. .......................................................... T F

(l) The Chernoff bound we learned in class is useful for getting lower bounds on the tails of a binomial distribution .......................................................... T F

(m) If $Y$ and $Z$ are independent random variables, then so are $f(Y)$ and $g(Z)$ for any functions $f$ and $g$. ($f$ and $g$ are functions of a single variable.) .......................................................... T F

(n) Let $f(x, y)$ be the joint density function of two random variables $X$ and $Y$. Then $F_X(a) = \int_{-\infty}^{a} f(x, y)dy$. .......................................................... T F
2. (27 points)

Short answer: (Don’t bother simplifying any of your answers.)

(a) An urn contains 5 white and 10 black balls. A fair 6-sided die is rolled and that number of balls are randomly selected without replacement from the urn. (So, for example, if the die shows a 4, then 4 balls will be selected from the urn.) What is the probability that all of the balls selected are white?

(b) An urn contains 5 white and 10 black balls. A fair die is rolled and that number of balls is randomly selected without replacement from the urn. What is the expected number of white balls selected?

(c) An urn contains 5 white and 10 black balls. A fair die is rolled and that number of balls is randomly selected without replacement from the urn. What is the conditional probability that the die landed on 3 if all of the balls selected are white?
An instructor gives a test consisting of 100 true/false problems. Each correct answer is worth 6 points and for each incorrect answer 4 points are deducted from the total score.

(a) What is the expected test score of a student that guesses independently and randomly on every single question? The student is equally likely to guess true or false.

(b) Same setup as the previous part. What is the variance in the total test score of a student who guesses independently on each question?

(c) Suppose that the student knows the answer to each question (independently) with probability 1/4. If he knows the answer, he just answers the question correctly. If he doesn’t know the answer he guesses (and as before, he is equally likely to guess true or false). What is the probability that the student gets $k$ questions right?

(d) Same scenario as previous part: the student knows the answer to each question (independently) with probability 1/4. If he knows the answer, he just answers the question correctly. If he doesn’t know the answer he guesses (and in this case, he is equally likely to guess true or false). What is his expected test score?
4. (30 points)

The probability density of $X$, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leq 10. \end{cases}$$

(a) What is $P_r(X > 30)$?

(b) What is the cumulative distribution function of $X$?

(c) What is the probability that of 6 (independent) devices of this type at least 4 will function for at least 30 hours? (There is no need to simplify your answer.)
6. (50 points)

(For parts (a) and (b) of this problem plug in numbers from the standard normal table on the first page of this test. But once you have done that, there is no need to simplify or calculate your answer.)

Let $X_1, \ldots, X_{100}$ be independent normal random variables, each with mean 3 and variance 36.

(a) (10 points) What is the $Pr(-21 \leq X_1 \leq 9)$?

(b) (10 points) What is the $Pr(\sum_{i=1}^{100} X_i \leq 60)$?

(c) (10 points) Suppose that $X$ is normal with mean $\mu$ and variance $\sigma^2$. What is the mean and variance of $Y = aX + b$ where $a$ and $b$ are constants.

(d) (20 points) Prove that $Y$ from the previous part is a normal random variable. Assume that $a > 0$. (Hint: Find the distribution function of $Y$, then differentiate to get density function and then show it has the appropriate form.)
6. (30 points)

The number of searches performed on Google each millisecond\(^1\) is a random variable with mean 4.

(a) Use the Markov Inequality to give an upper bound on the probability that at least 5 searches are performed each millisecond.

(b) Suppose it is known that the standard deviation of the number of searches is \(\sigma = 0.1\). Give an improved upper bound on the probability that at least 5 searches are performed each millisecond.

(c) Suppose that the number of searches performed each millisecond are independent, identically distributed random variables with mean 4 and standard deviation 0.1. What bound does the Central Limit Theorem give you on the probability that at least 410 searches are performed in a 100 milliseconds?

\(^1\)These numbers are completely made up.
7. (25 points)

Assume that $X_1, X_2, \ldots$ are i.i.d. random variables sampled from a distribution with mean $0 < \mu < \infty$ and variance $0 < \sigma^2 < \infty$. Circle the best single answer, one of (i) through (v), for each question.

(a) What is $S_n = \mathbb{E}[X_1 + X_2 + \cdots + X_n]$? i. 0 ii. $\mu/n$ iii. $\mu/\sqrt{n}$ iv. $\mu$ v. $n\mu$

(b) What is $T_n = \mathbb{E}[(X_1 + X_2 + \cdots + X_n)/n]$? i. 0 ii. $\mu/n$ iii. $\mu/\sqrt{n}$ iv. $\mu$ v. $n\mu$

(c) What is $U_n = \text{Var}[X_1 + X_2 + \cdots + X_n]$? i. $\sigma^2/n^2$ ii. $\sigma^2/n$ iii. $\sigma^2$ iv. $n\sigma^2$ v. $n^2\sigma^2$

(d) What is $V_n = \text{Var}[(X_1 + X_2 + \cdots + X_n)/n]$? i. $\sigma^2/n^2$ ii. $\sigma^2/n$ iii. $\sigma^2$ iv. $n\sigma^2$ v. $n^2\sigma^2$

(e) In the limit as $n \to \infty$,
   i. All 4 of $S_n, T_n, U_n, V_n$ go to infinity
   ii. All 4 go to zero
   iii. $S_n$ and $U_n$ go to infinity, while $T_n$ and $V_n$ converge to $\mu$ and $\sigma^2$, respectively
   iv. $S_n$ and $U_n$ go to infinity, $V_n$ goes to zero, while $T_n$ remains constant, independent of $n$
   v. none of the above
8. (26 points) In this problem you do not need to verify the second order conditions.

(a) Let \( x_1, x_2, \ldots, x_n \) be \( n \) samples from a Poisson distribution with unknown parameter \( \theta \) (otherwise known as \( \lambda \)). What is the maximum likelihood estimate for \( \theta \)?
(b) Individuals in a certain country are voting in an election between three candidates, A, B and C. Suppose that independently each person votes for candidate A with probability $p$, for candidate B with probability $r$ and for candidate C with probability $1 - p - r$. The parameters $p$ and $r$ are unknown. Suppose that $X_1, \ldots, X_n$ are $n$ independent, identically distributed votes from this distribution. (Let $x_A =$ number of $X_i$'s equal to A, let $x_B =$ number of $X_i$'s equal to B, and let $x_C =$ number of $X_i$'s equal to C.)

i. (5 points) What is the likelihood function $L(x_A, x_B, x_C | p, r)$?

ii. (4 points) What is the log-likelihood function?

iii. (15 points) Find the maximum likelihood estimates $\hat{p}$ and $\hat{r}$. (You will get most of the points just for setting up the system of equations that need to be solved to find $\hat{p}$ and $\hat{r}$.)