independence

Defn: Two events E and F are *independent* if

$P(EF) = P(E)\ P(F)$

If $P(F)>0$, this is equivalent to: $P(E|F) = P(E)$ (proof below)

Otherwise, they are called *dependent*

Roll two dice, yielding values D_1 and D_2

1) $E = \{ D_1 = 1 \}$

$F = \{ D_2 = 1 \}$

$P(E) = 1/6$, $P(F) = 1/6$, $P(EF) = 1/36$

$P(EF) = P(E) \cdot P(F) \Rightarrow$ E and F *independent*

Intuitive; the two dice are not physically coupled

2) $G = \{D_1 + D_2 = 5\} = \{(1,4),(2,3),(3,2),(4,1)\}$

$P(E) = 1/6$, $P(G) = 4/36 = 1/9$, $P(EG) = 1/36$

not independent!

E, G are *dependent* events

The dice are still not physically coupled, but "$D_1 + D_2 = 5$" couples them *<u>mathematically:</u> info about D_1 constrains D_2. (But dependence/independence not always intuitively obvious; "use the definition, Luke".)*

Two events E and F are *independent* if

$P(EF) = P(E)\,P(F)$

If $P(F)>0$, this is equivalent to: $P(E|F) = P(E)$

Otherwise, they are called *dependent*

Three events E, F, G are independent if

$$P(EF) = P(E)\,P(F)$$
$$P(EG) = P(E)\,P(G) \quad \textit{and} \quad P(EFG) = P(E)\,P(F)\,P(G)$$
$$P(FG) = P(F)\,P(G)$$

Example: Let X, Y be each $\{-1,1\}$ all outcomes equally likely

$E = \{X = 1\}$, $F = \{Y = 1\}$, $G = \{XY = 1\}$

$P(EF) = P(E)P(F)$, $P(EG) = P(E)P(G)$, $P(FG) = P(F)P(G)$

but $P(EFG) = 1/4$!!

In general, events $E_1, E_2, \ldots, E_n$ are independent if for *every subset* S of {1,2,…, n}, we have

$$P\left(\bigcap_{i \in S} E_i\right) = \prod_{i \in S} P(E_i)$$

(Sometimes this property holds only for small subsets S. E.g., E, F, G on the previous slide are *pairwise* independent, but not fully independent.)

Theorem: E, F independent $\Rightarrow$ E, F^c independent

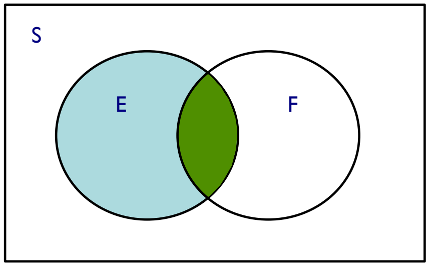
Proof: $P(EF^c) = P(E) - P(EF)$

$= P(E) - P(E)\,P(F)$

$= P(E)\,(1-P(F))$

$= P(E)\,P(F^c)$

$E = EF \cup EF^c$



Theorem: $P(E)>0$, $P(F)>0$

E, F independent $\Leftrightarrow P(E|F)=P(E) \Leftrightarrow P(F|E) = P(F)$

Proof: Note $P(EF) = P(E|F)\,P(F)$, regardless of in/dep.

Assume independent. Then

$P(E)P(F) = P(EF) = P(E|F)\,P(F) \Rightarrow P(E|F)=P(E)$ (÷ by P(F))

Conversely, $P(E|F)=P(E) \Rightarrow P(E)P(F) = P(EF)$ (× by P(F))

Suppose a biased coin comes up heads with probability p, *independent* of other flips

P(n heads in n flips) = p^n

P(n tails in n flips) = $(1-p)^n$

P(exactly k heads in n flips) $= \binom{n}{k} p^k (1-p)^{n-k}$

Aside: note that the probability of *some* number of heads = $\sum_k \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1$

as it should, by the binomial theorem.

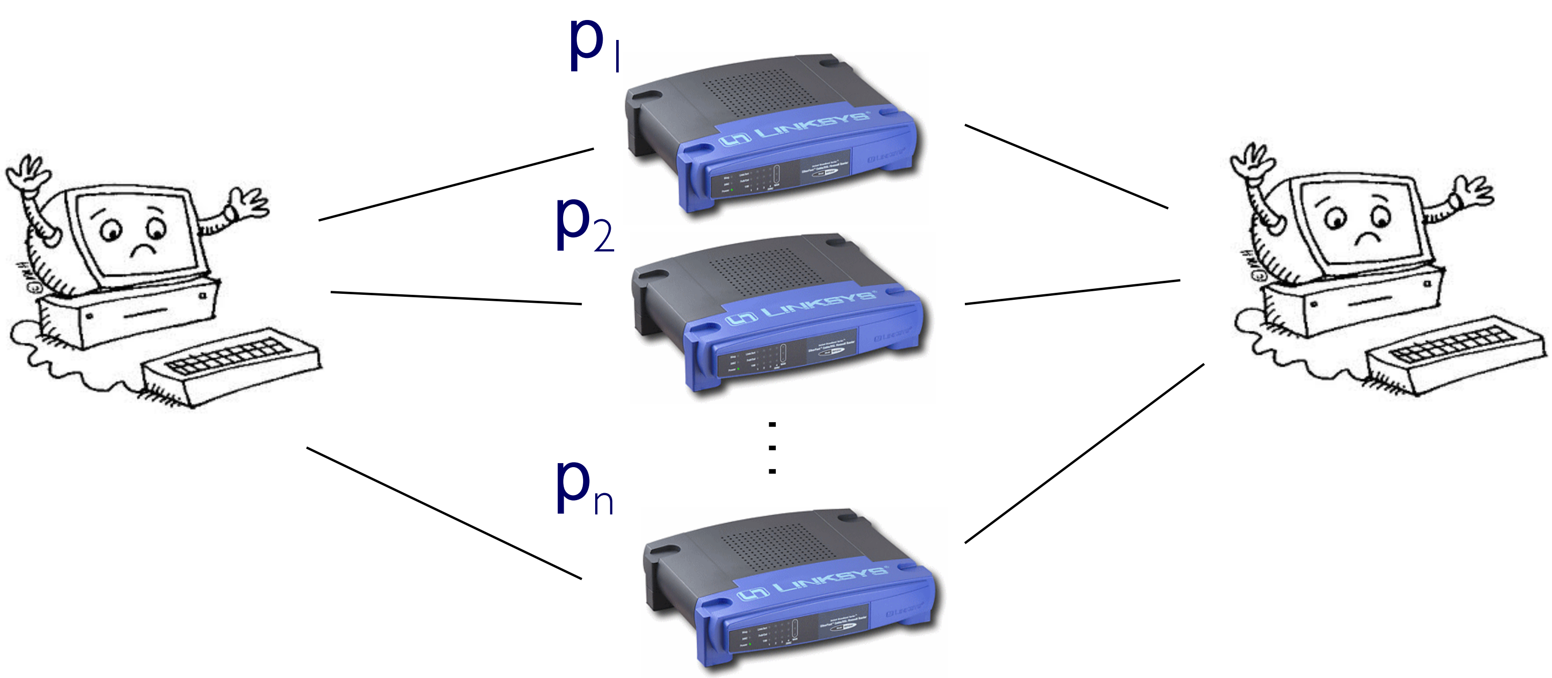
Suppose a biased coin comes up heads with probability p, *independent* of other flips

P(exactly k heads in n flips) $= \binom{n}{k} p^k (1-p)^{n-k}$

Note when $p=1/2$, this is the same result we would have gotten by considering n flips in the "equally likely outcomes" scenario. But $p \neq 1/2$ makes that inapplicable. Instead, the *independence* assumption allows us to conveniently assign a probability to each of the 2^n outcomes, e.g.:

$\Pr(\text{HHTHTTT}) = p^2(1-p)p(1-p)^3 = p^{\#H}(1-p)^{\#T}$

Consider the following parallel network

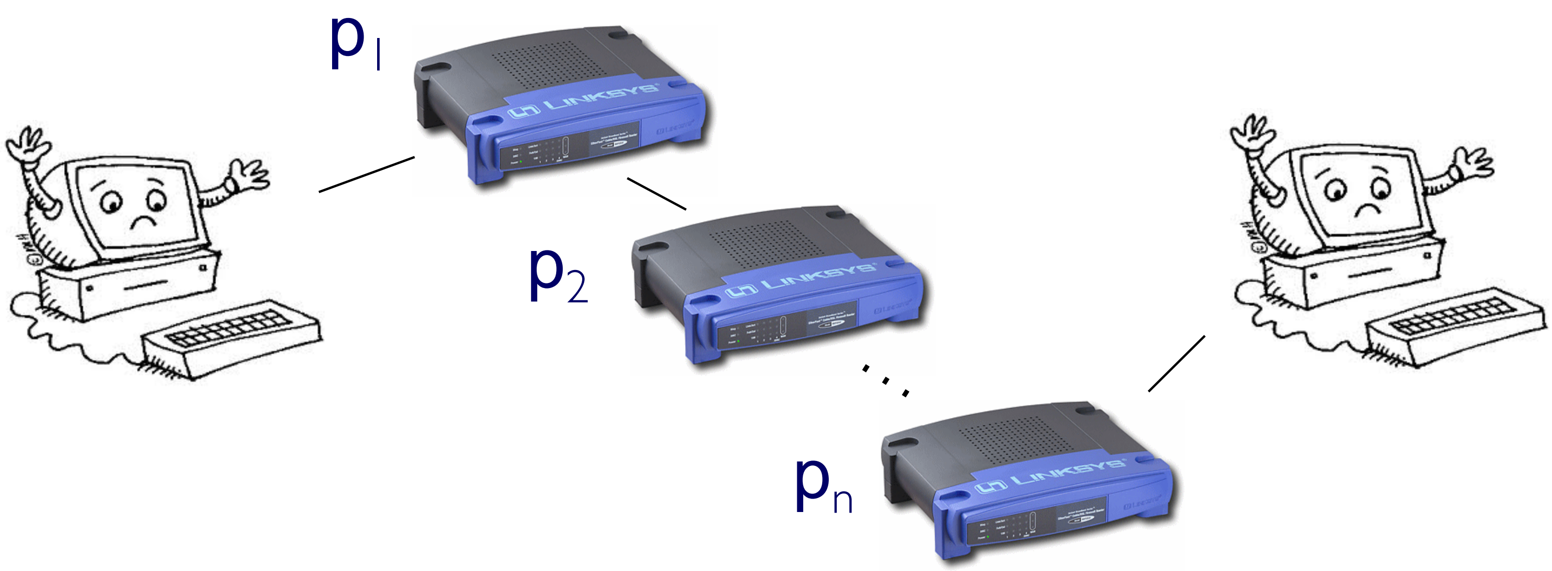


n routers, i^{th} has probability p_i of failing, independently

P(there is functional path) = 1 – P(all routers fail)

$$= 1 - p_1 p_2 \cdots p_n$$

Contrast: a *series* network



n routers, i^{th} has probability p_i of failing, independently

P(there is functional path) =

$$P(\textit{no} \text{ routers fail}) = (1 - p_1)(1 - p_2) \cdots (1 - p_n)$$