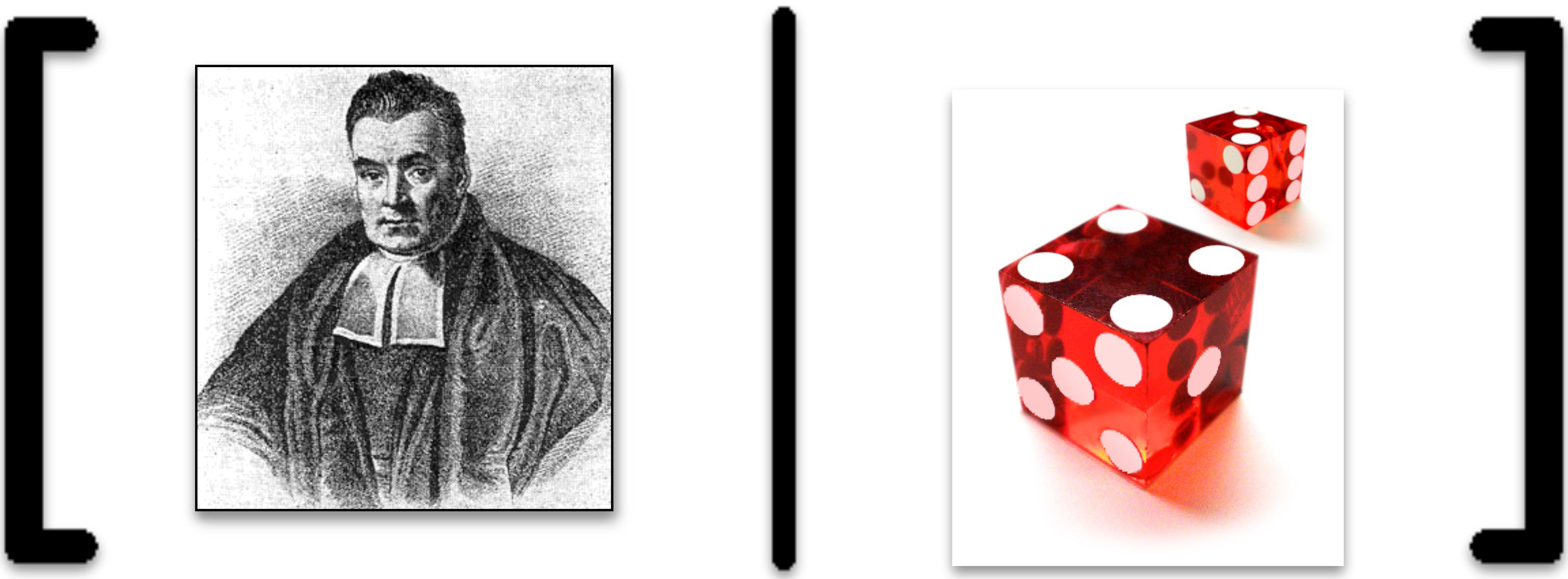
conditional probability

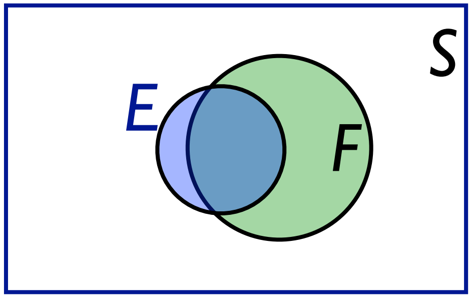


Conditional probability of E given F: probability that E occurs *given* that F has occurred.

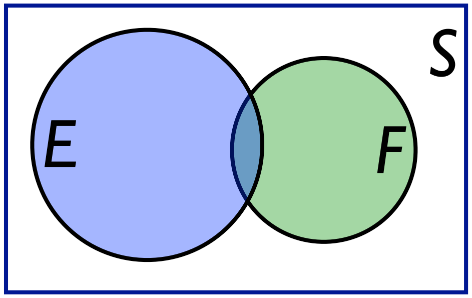
"Conditioning on F"

Written as P(E|F)

Means "P(E, given F observed)"



Sample space S reduced to those elements consistent with F (i.e. $S \cap F$)



Event space E reduced to those elements consistent with F (i.e. $E \cap F$)

With equally likely outcomes,

$$P(E \mid F) = \frac{\#\text{ of outcomes in } E \text{ consistent with } F}{\#\text{ of outcomes in } S \text{ consistent with } F} = \frac{|EF|}{|SF|} = \boxed{\frac{|EF|}{|F|}}$$

$$P(E \mid F) = \frac{|EF|}{|F|} = \frac{|EF|/|S|}{|F|/|S|} = \boxed{\frac{P(EF)}{P(F)}}$$

Suppose you flip two coins & all outcomes are equally likely.

What is the probability that both flips land on heads if…

- The first flip lands on heads?

  Let B = {HH} and F = {HH, HT}

  P(B|F) = P(BF)/P(F) = P({HH})/P({HH, HT})

  = (1/4)/(2/4) = 1/2

- At least one of the two flips lands on heads?

  Let A = {HH, HT, TH}, BA = {HH}

  P(B|A) = |BA|/|A| = 1/3

- At least one of the two flips lands on tails?

  Let G = {TH, HT, TT}

  P(B|G) = P(BG)/P(G) = P(∅)/P(G) = 0/P(G) = 0

General defn: $$P(E \mid F) = \frac{P(EF)}{P(F)}$$ where P(F) > 0

Holds even when outcomes are *not* equally likely.

What if P(F) = 0?

P(E|F) undefined: (you can't observe the impossible)

Implies: P(EF) = P(E|F) P(F) *("the chain rule")*

General definition of Chain Rule:

$$P(E_1 E_2 \cdots E_n) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1, E_2) \cdots P(E_n \mid E_1, E_2, \ldots, E_{n-1})$$

General defn: $$P(E \mid F) = \frac{P(EF)}{P(F)}$$ where P(F) > 0

Holds even when outcomes are *not* equally likely.

"P(- | F)" is a probability law, i.e. satisfies the 3 axioms

Proof:

the idea is simple–the sample space contracts to F; dividing all (unconditional) probabilities by P(F) correspondingly re-normalizes the probability measure – see text for details; better yet, try it!

Ex: $P(A \cup B) \leq P(A) + P(B)$

$\therefore P(A \cup B \mid F) \leq P(A \mid F) + P(B \mid F)$

sending bit strings

Bit string with m 0's and n 1's sent on the network

All distinct arrangements of bits equally likely

E = first bit received is a 1

F = k of first r bits received are 1's

What's P(E|F)?

Solution 1:

$$P(E) = \frac{n}{m+n} \qquad P(F) = \frac{\binom{n}{k}\binom{m}{r-k}}{\binom{m+n}{r}}$$

$$P(F \mid E) = \frac{\binom{n-1}{k-1}\binom{m}{r-k}}{\binom{m+n-1}{r-1}}$$

$$P(E \mid F) = \frac{P(EF)}{P(F)} = \frac{P(F \mid E)P(E)}{P(F)} = \frac{k}{r}$$

Bit string with m 0's and n 1's sent on the network

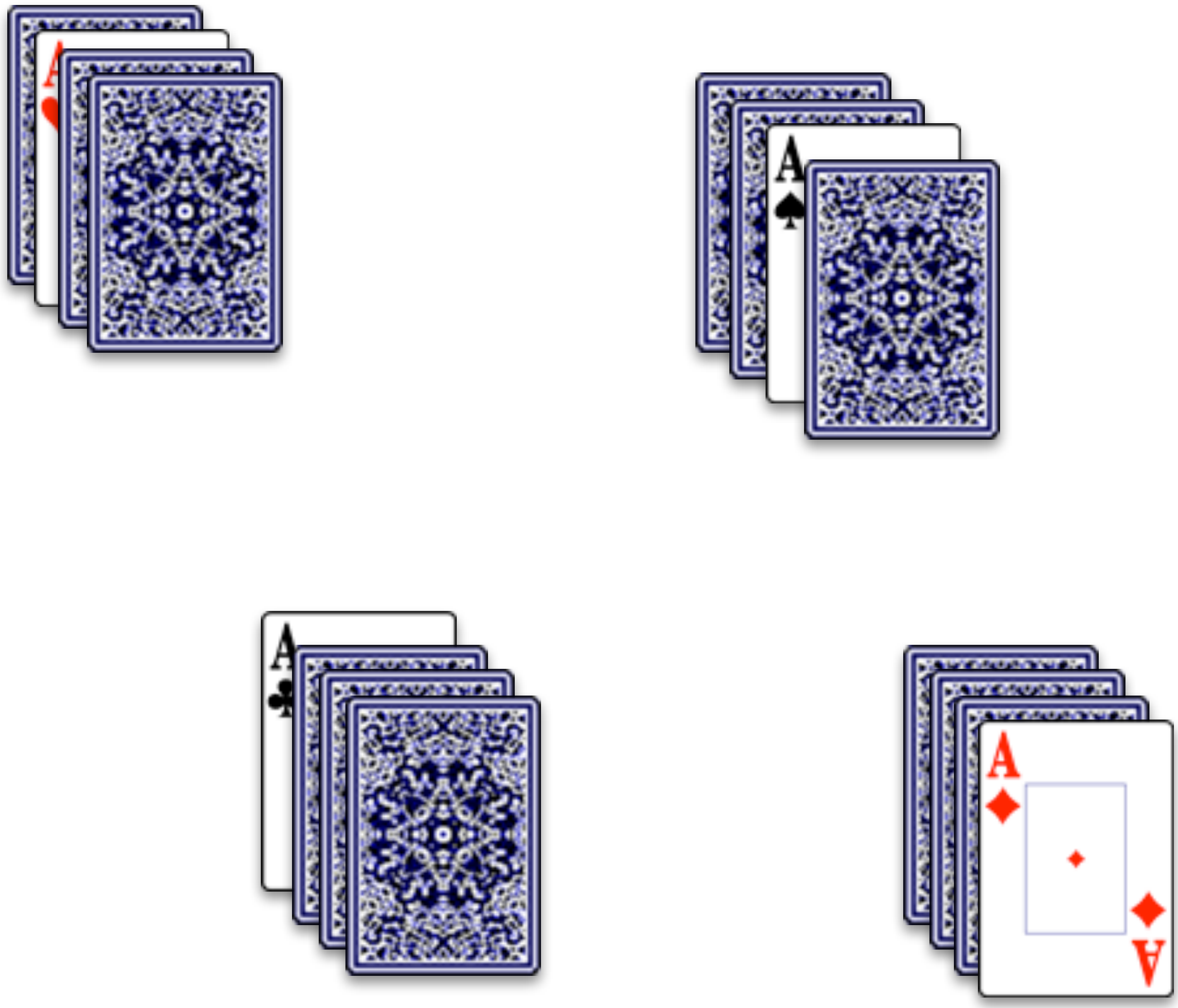
- All distinct arrangements of bits equally likely
- E = first bit received is a 1
- F = k of first r bits received are 1's

What's P(E|F)?

Solution 2:

- Observe:
  - P(E|F) = P(picking one of k 1's out of r bits)
- So:
  - P(E|F) = k/r

piling cards



piling cards

Deck of 52 cards randomly divided into 4 piles

13 cards per pile

Compute P(each pile contains an ace)

Solution:

E_1 = {  in some pile }

E_2 = {  &  in different piles }

E_3 = {    in different piles }

E_4 = { all four aces in different piles }

Compute $P(E_1 E_2 E_3 E_4)$

piling cards

E_1 = {  in some pile }

E_2 = {  &  in different piles }

E_3 = {    in different piles }

E_4 = { all four aces in different piles }

$$P(E_1 E_2 E_3 E_4)$$
$$= P(E_1)\ P(E_2|E_1)\ P(E_3|E_1 E_2)\ P(E_4|E_1 E_2 E_3)$$

piling cards

E_1 = {  in any one pile }

E_2 = {  &  in different piles }

E_3 = {    in different piles }

E_4 = { all four aces in different piles }

$P(E_1E_2E_3E_4) = P(E_1)\ P(E_2|E_1)\ P(E_3|E_1E_2)\ P(E_4|E_1E_2E_3)$

$P(E_1) = 1$

$P(E_2|E_1) = 39/51$ (39 of 51 slots not in AH pile)

$P(E_3|E_1E_2) = 26/50$ (26 not in AS, AH piles)

$P(E_4|E_1E_2E_3) = 13/49$ (13 not in AS, AH, AD piles)

A conceptual trick: what's randomized?

a) *randomize* cards, deal *sequentially* into piles
b) *sort* cards, aces first, deal *randomly* into piles.

piling cards

E_1 = {  in any one pile }

E_2 = {  &  in different piles }

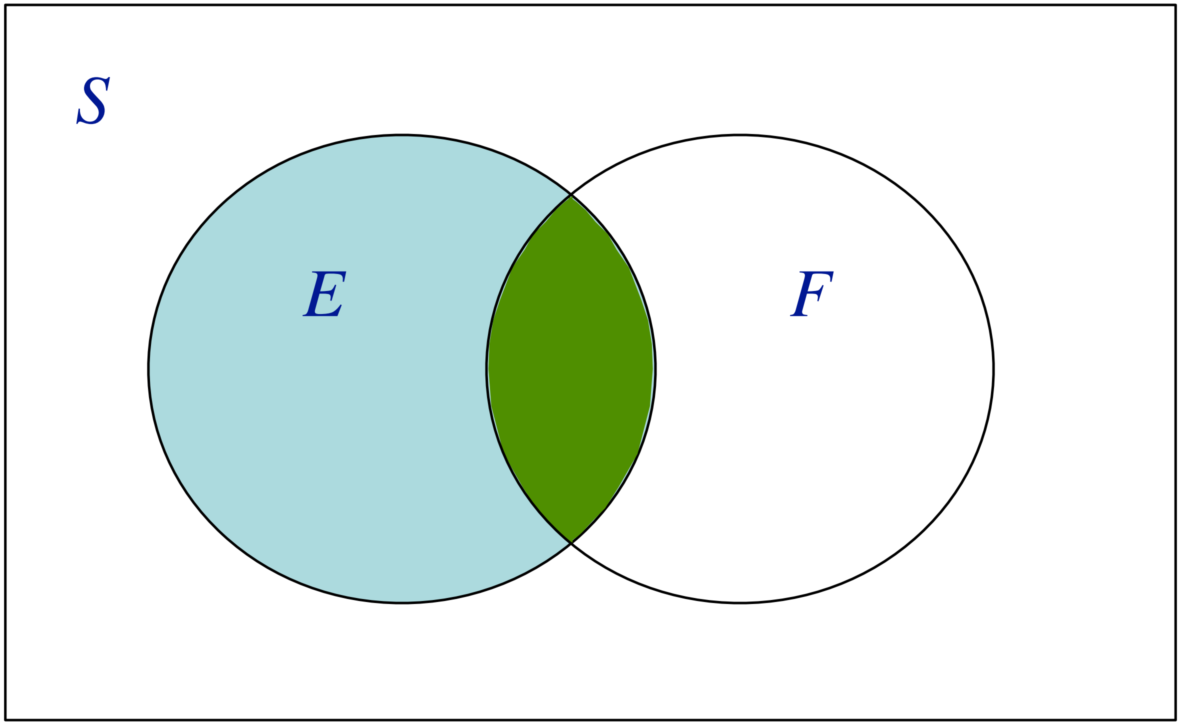
E_3 = {    in different piles }

E_4 = { all four aces in different piles }

$$P(E_1E_2E_3E_4)$$
$$= P(E_1)\ P(E_2|E_1)\ P(E_3|E_1E_2)\ P(E_4|E_1E_2E_3)$$
$$= (39 \cdot 26 \cdot 13)/(51 \cdot 50 \cdot 49)$$
$$\approx 0.105$$

E and F are events in the sample space S

$$E = EF \cup EF^c$$



$$EF \cap EF^c = \varnothing$$

$$\Rightarrow P(E) = P(EF) + P(EF^c)$$

law of total probability

$$P(E) = P(EF) + P(EF^c)$$
$$= P(E|F)\,P(F) + P(E|F^c)\,P(F^c)$$
$$= P(E|F)\,P(F) + P(E|F^c)\,(1-P(F))$$

weighted average, conditioned on event F happening or not.

More generally, if $F_1, F_2, \ldots, F_n$ partition S (mutually exclusive, $\bigcup_i F_i = S$, $P(F_i)>0$), then

$$P(E) = \sum_i P(E|F_i)\,P(F_i)$$

weighted average, conditioned on events F_i happening or not.

Example

Pick a random ball from one of 3 boxes

box 1, prob 1/2: 5 red balls 5 green balls

box 2, prob 1/3: 1 red ball 6 green balls

box 3, prob 1/6: 8 red balls 4 green balls

What is the probability of picking a green ball?

What is the probability that we chose from box 2 given that we picked a green ball?