4: Discrete probability

Readings: BT 1.1-1.2, Rosen 6.1-6.2

Sample space: *S* is the set of all possible outcomes of an experiment (Ω in your text book–Greek uppercase omega)

Coin flip:	S = {Heads, Tails}
Flipping two coins:	S = {(H,H), (H,T), (T,H), (T,T)}
Roll of one 6-sided die:	S = {1, 2, 3, 4, 5, 6}
# emails in a day:	$S = \{ x : x \in Z,\ x \geq 0 \}$
YouTube hrs. in a day:	$S = \{ x : x \in R, 0 \leq x \leq 24 \}$

Events: **E** $\subseteq$ **S** is some subset of the sample space

Coin flip is heads:	E = {Head}
At least one head in 2 flips:	E = {(H,H), (H,T), (T,H)}
Roll of die is 3 or less:	E = {1, 2, 3}
# emails in a day < 20:	$E = \{ x : x \in Z,\ 0 \leq x < 20\}$
Wasted day (>5 YT hrs):	$E = \{ x : x \in R,\ x > 5 \}$

Intuition: Probability as the relative frequency of an event

$\Pr(E) = \lim_{n \to \infty}$ (# of occurrences of E in n trials)/n

Axiom 1: $0 \leq \Pr(E) \leq 1$

Axiom 2: $\Pr(S) = 1$

Axiom 3: If E and F are mutually exclusive ($EF = \emptyset$), then

$$\Pr(E \cup F) = \Pr(E) + \Pr(F)$$

For any sequence $E_1, E_2, \ldots, E_n$ of mutually exclusive events,

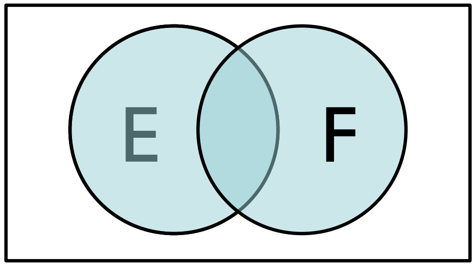
$$\Pr\left(\bigcup_{i=1}^{n} E_i\right) = \Pr(E_1) + \cdots + \Pr(E_n)$$

- $\Pr(\overline{E}) = 1 - \Pr(E)$

- If $E \subseteq F$, then $\Pr(E) \leq \Pr(F)$

- $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(EF)$

- And many others



equally likely outcomes

Simplest case: sample spaces with equally likely outcomes.

Coin flips:	S = {Heads, Tails}
Flipping two coins:	S = {(H,H),(H,T),(T,H),(T,T)}
Roll of 6-sided die:	S = {1, 2, 3, 4, 5, 6}

Pr(each outcome) = $\frac{1}{|S|}$

uniform distribution

In that case,

$$\Pr(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}$$

probability calculations can be slippery

Intuition can mislead you

4-step approach:

1. Find the sample space
2. Define events of interest
3. Determine outcome probabilities
4. Compute event probabilities

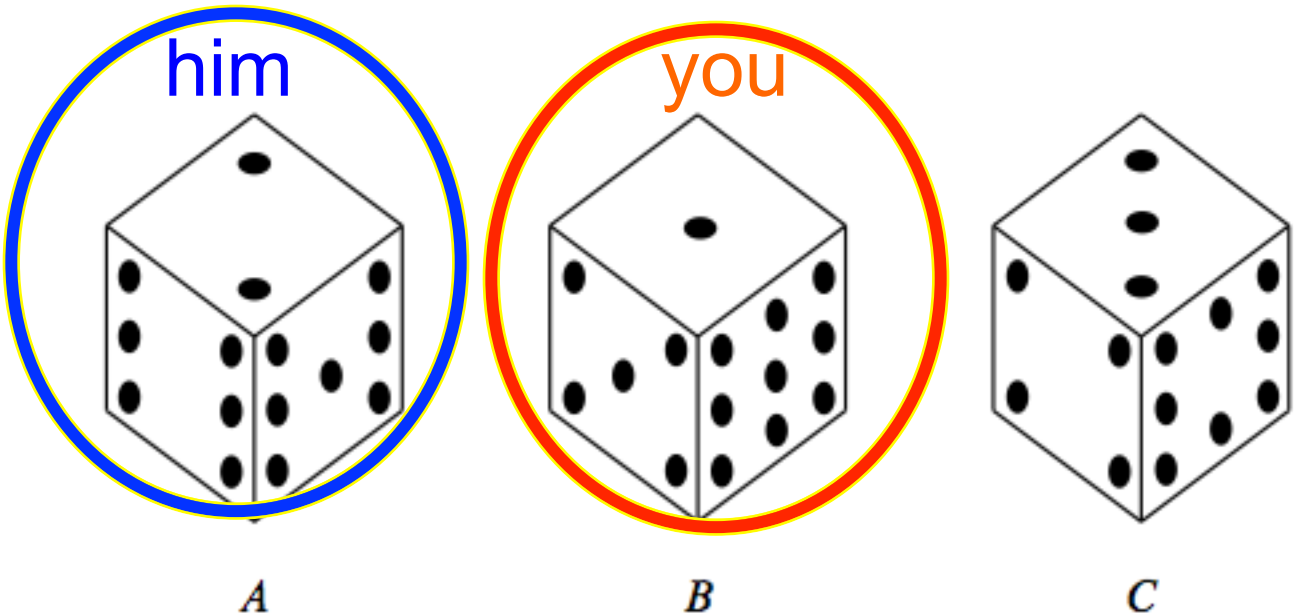
Strange dice

You go to a bar. The guy sitting next to you pulls out dice.

He offers you a $100 wager:

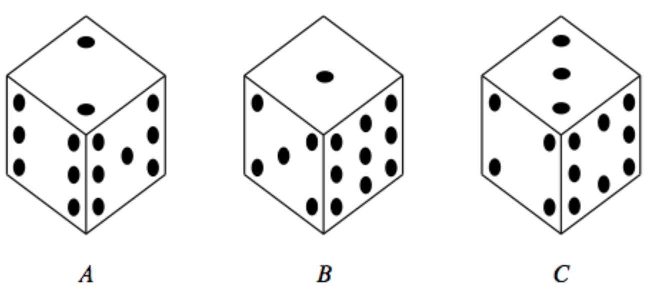
Each player selects one die and rolls it once.

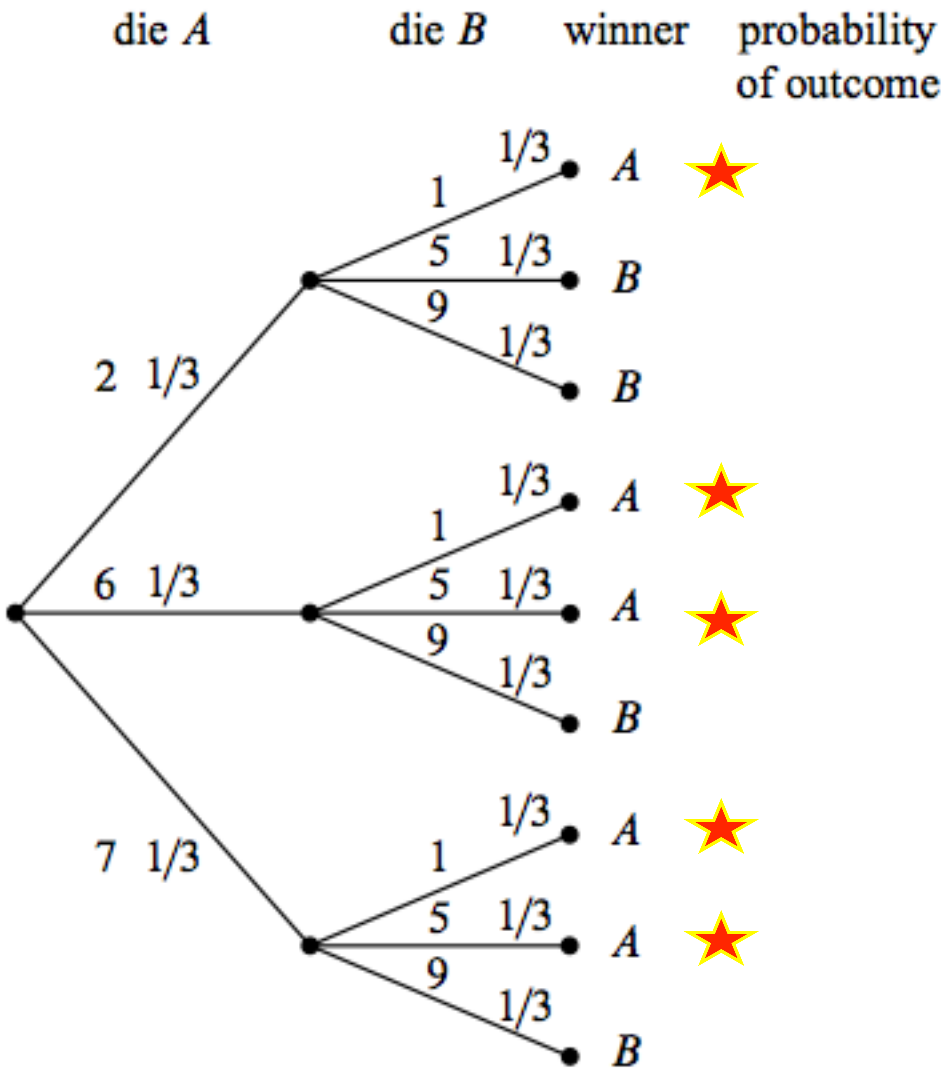
The player with the lower value pays the other player $100.



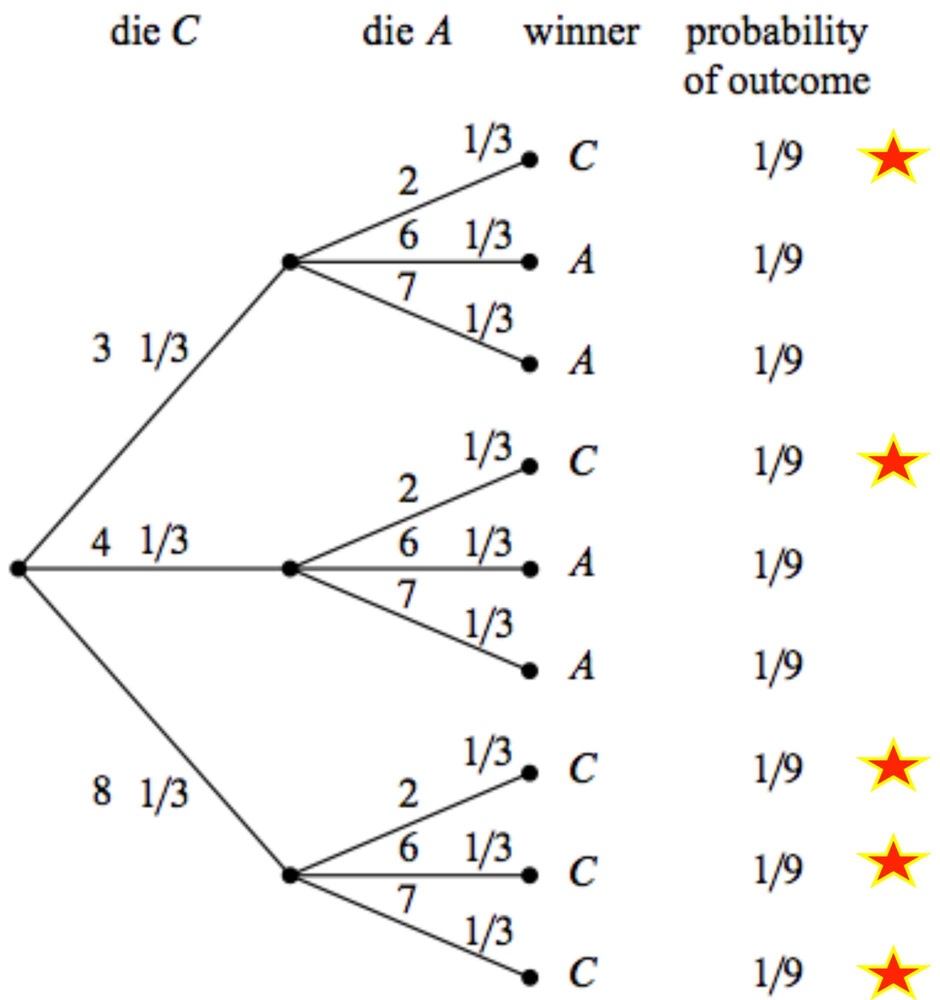
You picked B, he picked A

$A \succ B$



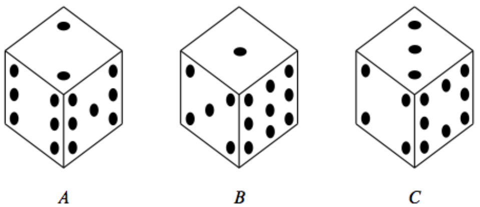


He gives you another chance: so you choose A



$C \succ A$

$A \succ B$



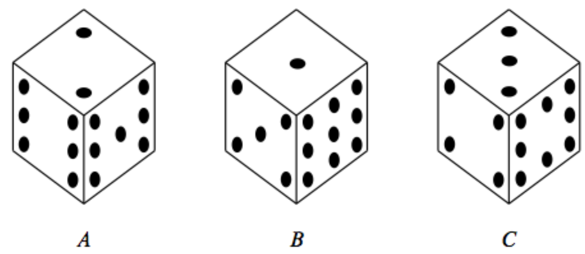
One final time: you get die C

Bizarre?

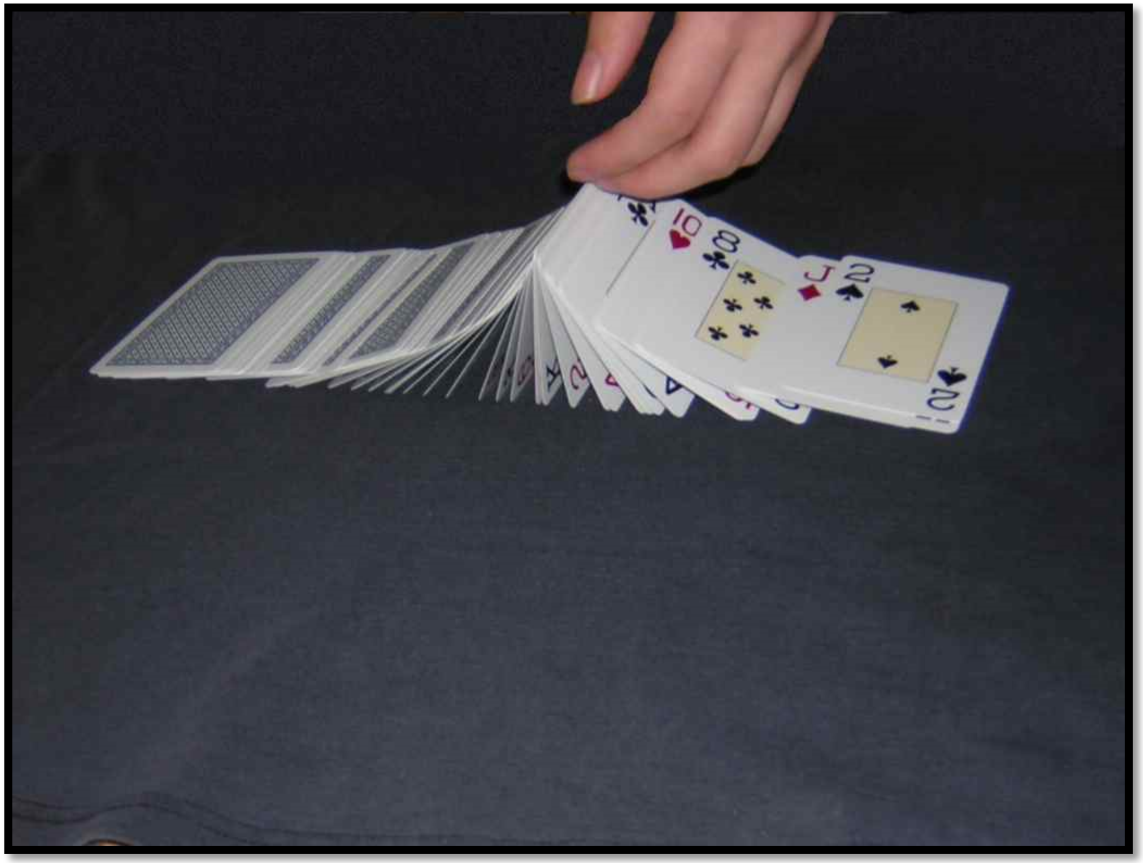
C beats A with probability $5/9$,

A beats B with probability $5/9$,

B beats C with probability $5/9$



card flipping



card flipping

52 card deck. Cards flipped one at a time.

After first ace (of any suit) appears, consider next card

Pr(next card = ace of spades) < Pr(next card = 2 of clubs) ?

Case 1: Take Ace of Spades out of deck

Shuffle remaining 51 cards, add ace of spades after first ace

|S| = 52! (all cards shuffled)

|E| = 51! (only 1 place ace of spades can be added)

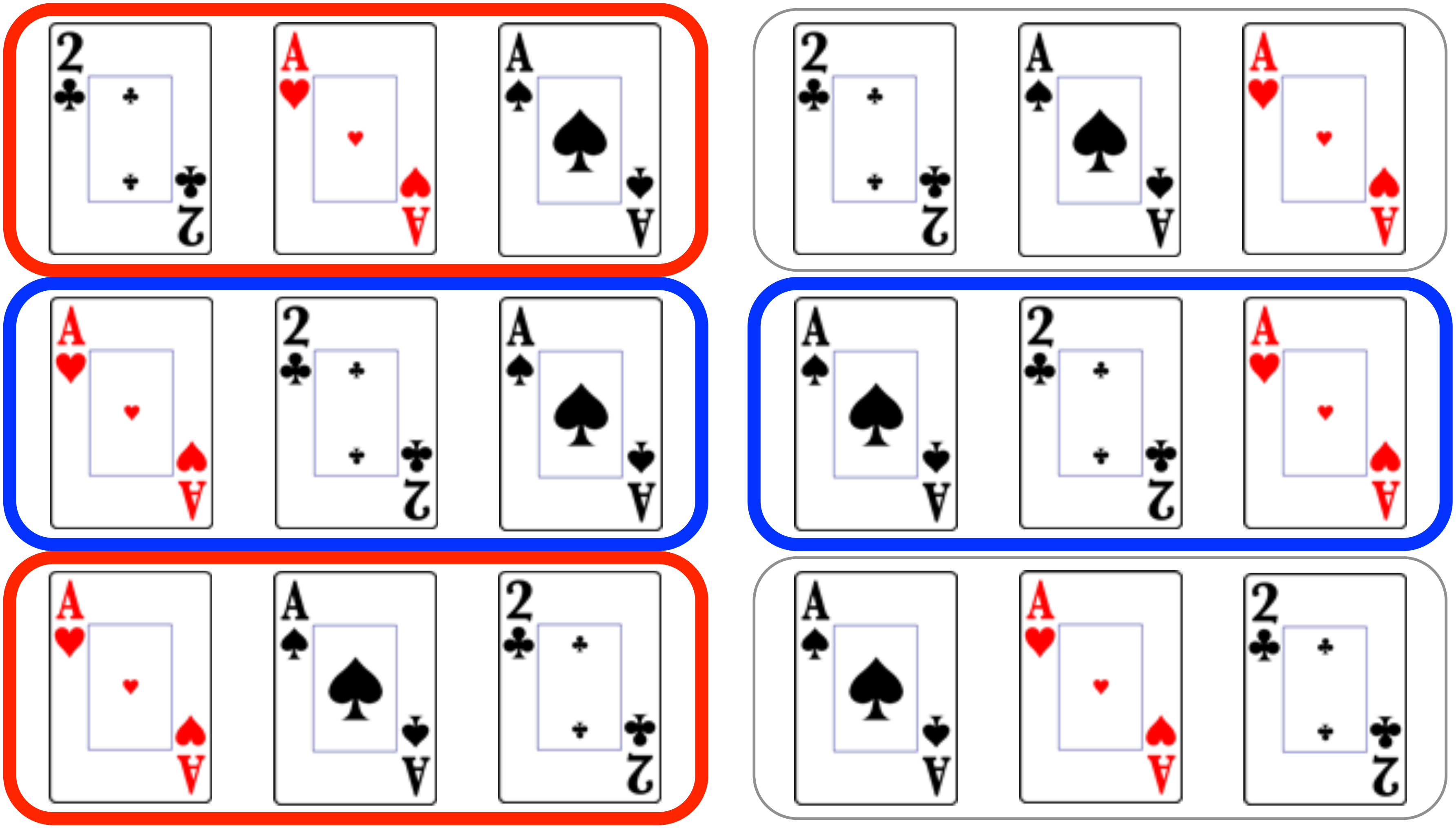
Case 2: Do the same thing with the 2 of clubs

|S| and |E| have same size

So,

Pr(next = Ace of spades) = Pr(next = 2 of clubs) = 1/52

Ace of Spades: 2/6 2 of Clubs: 2/6



Theory is the same for a 3-card deck; Pr = 2!/3! = 1/3

Card images from http://www.eludication.org/playingcards.html

hats

n persons at a party throw hats in middle, select at random. **What is Pr(no one gets own hat)?**

What is the sample space *S*?

People:	P_1	P_2	P_3	P_4	P_5
Hats:	H_4	H_2	H_5	H_1	H_3

I.e., a sample point is a *permutation* π of 1, …, n

4	2	5	1	3

$|S| = n!$

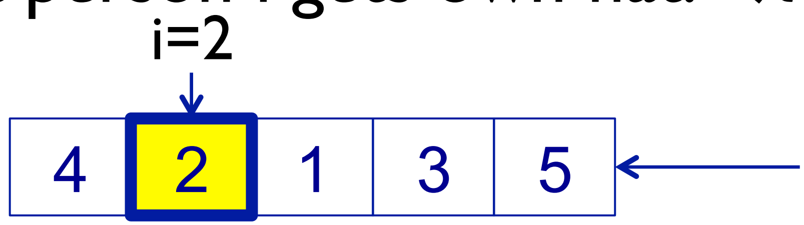
n persons at a party throw hats in middle, select at random. What is Pr(no one gets own hat)?

Pr(no one gets own hat) =
 1 – Pr(someone gets own hat)

Pr(someone gets own hat) = $\Pr(\cup_{i=1}^{n} E_i)$, where
E_i = event that person i gets own hat

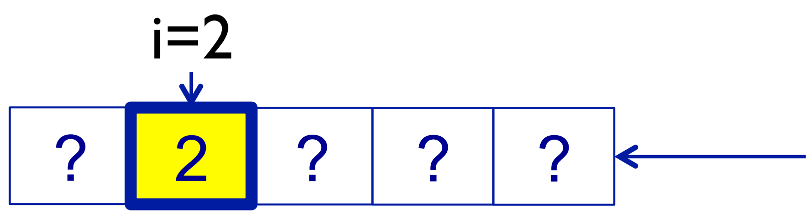
$\Pr(\cup_{i=1}^{n} E_i) = \sum_i P(E_i) - \sum_{i<j} \Pr(E_i E_j) + \sum_{i<j<k} \Pr(E_i E_j E_k) \ldots$

E_i = event that person i gets own hat: $\pi(i) = i$



A sample point in E_2 (also in E_5)

Counting single events:

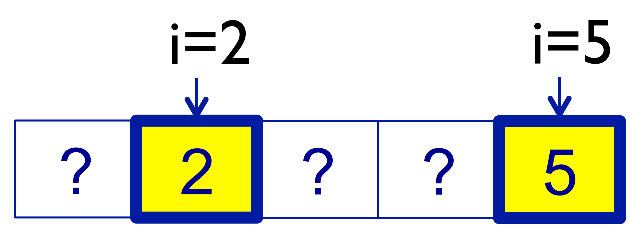


All points in E_2

$|E_i| = (n-1)!$ for all i

Counting pairs:

E_iE_j : $\pi(i) = i$ & $\pi(j) = j$



$|E_iE_j| = (n-2)!$ for all i, j

All points in $E_2 \cap E_5$

n persons at a party throw hats in middle, select at random. What is Pr(no one gets own hat)?

E_i = event that person i gets own hat

$$\Pr(\cup_{i=1}^{n} E_i) = \Sigma_i P(E_i) - \Sigma_{i<j} \Pr(E_i E_j) + \Sigma_{i<j<k} \Pr(E_i E_j E_k) \ldots$$

Pr(k fixed people get own back) = (n-k)!/n!

$$\binom{n}{k} \text{ times that } = \frac{n!}{k!(n-k)!} \frac{(n-k)!}{n!} = 1/k!$$

Pr(none get own) = 1-Pr(some do) =

$$1 - 1/1! + 1/2! - 1/3! + 1/4! \ldots + (-1)^n/n! \approx 1/e \approx .37$$

Pr(none get own) = 1 - Pr(some do) =

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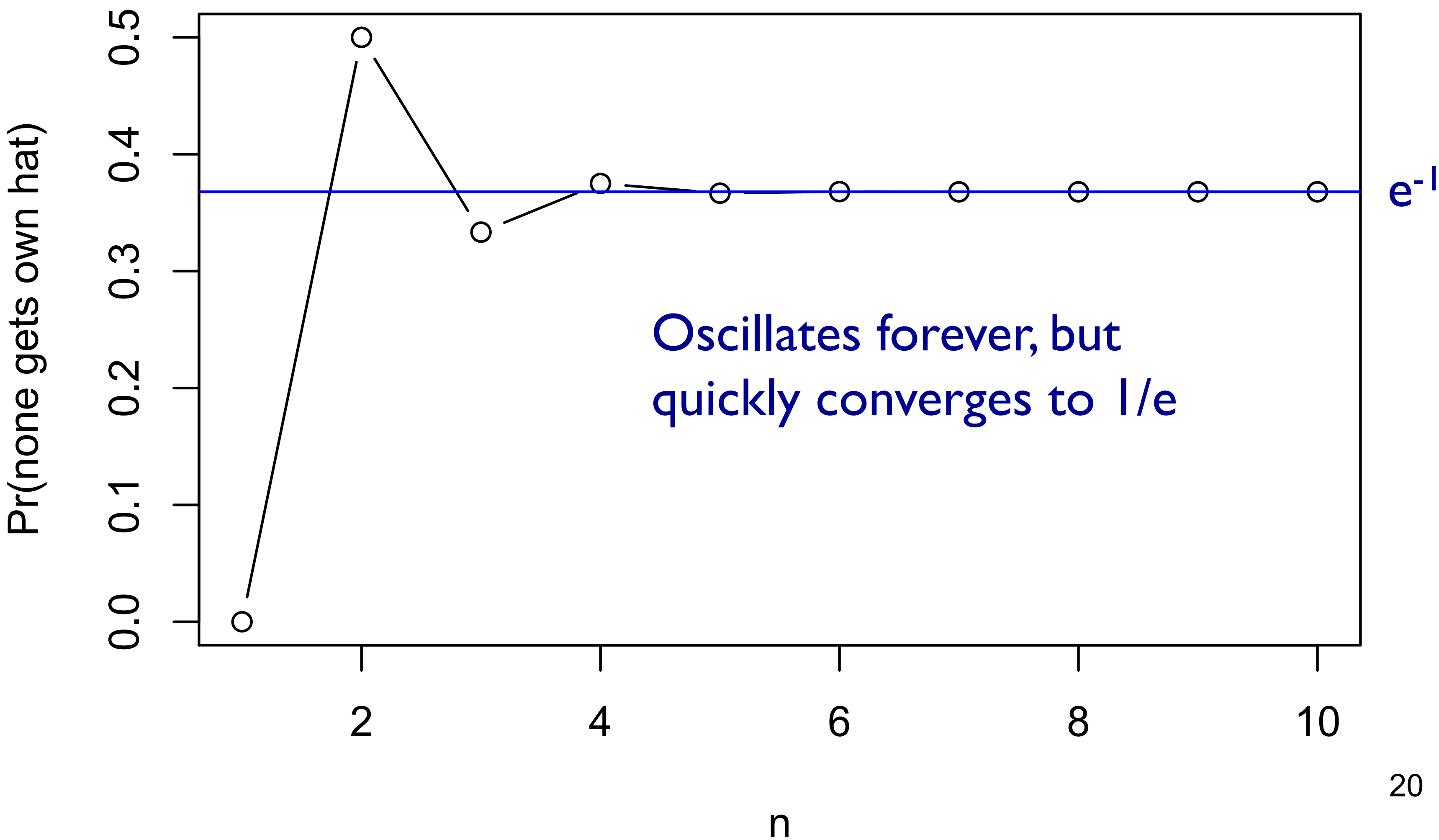
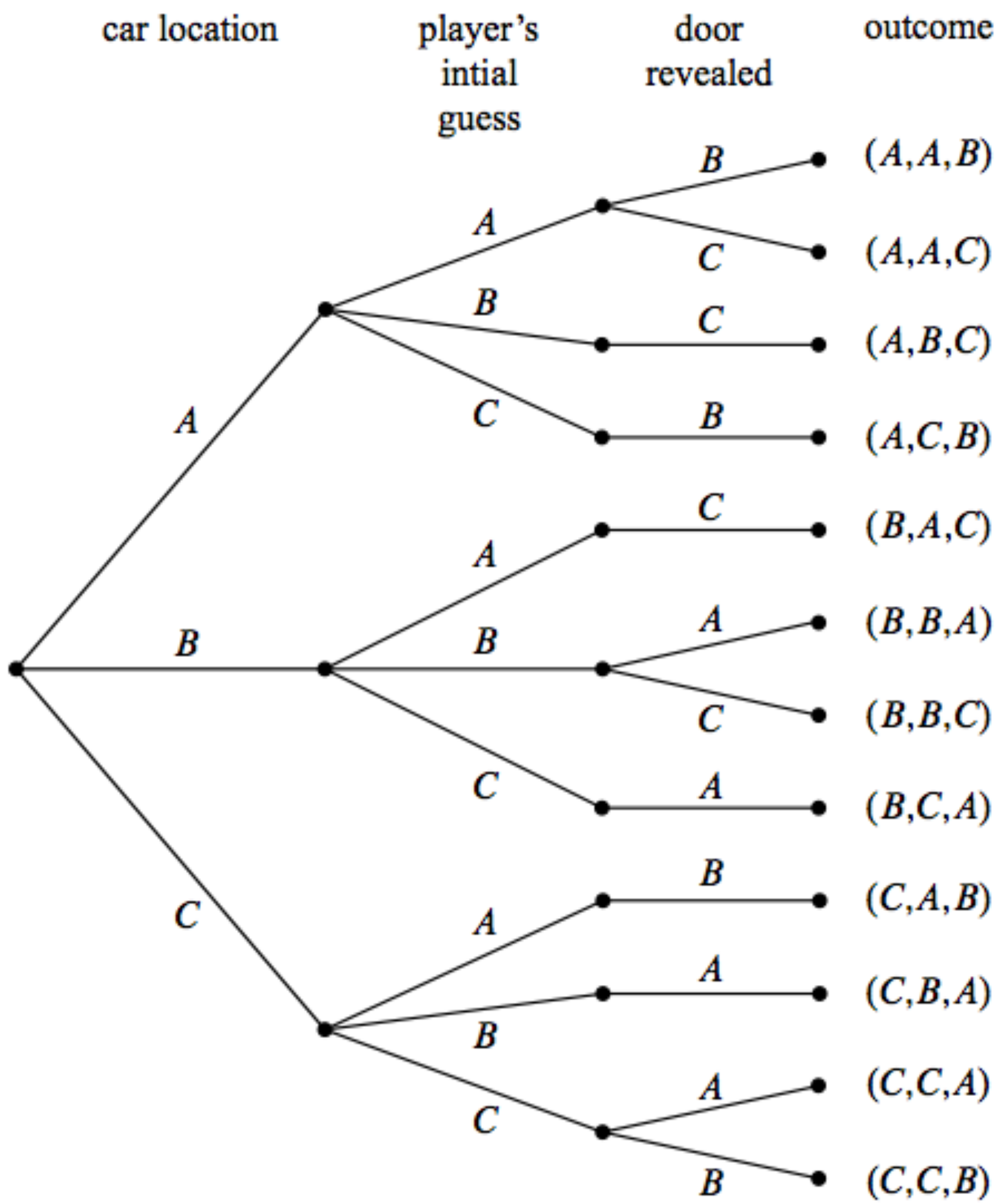


The Monty Hall Problem

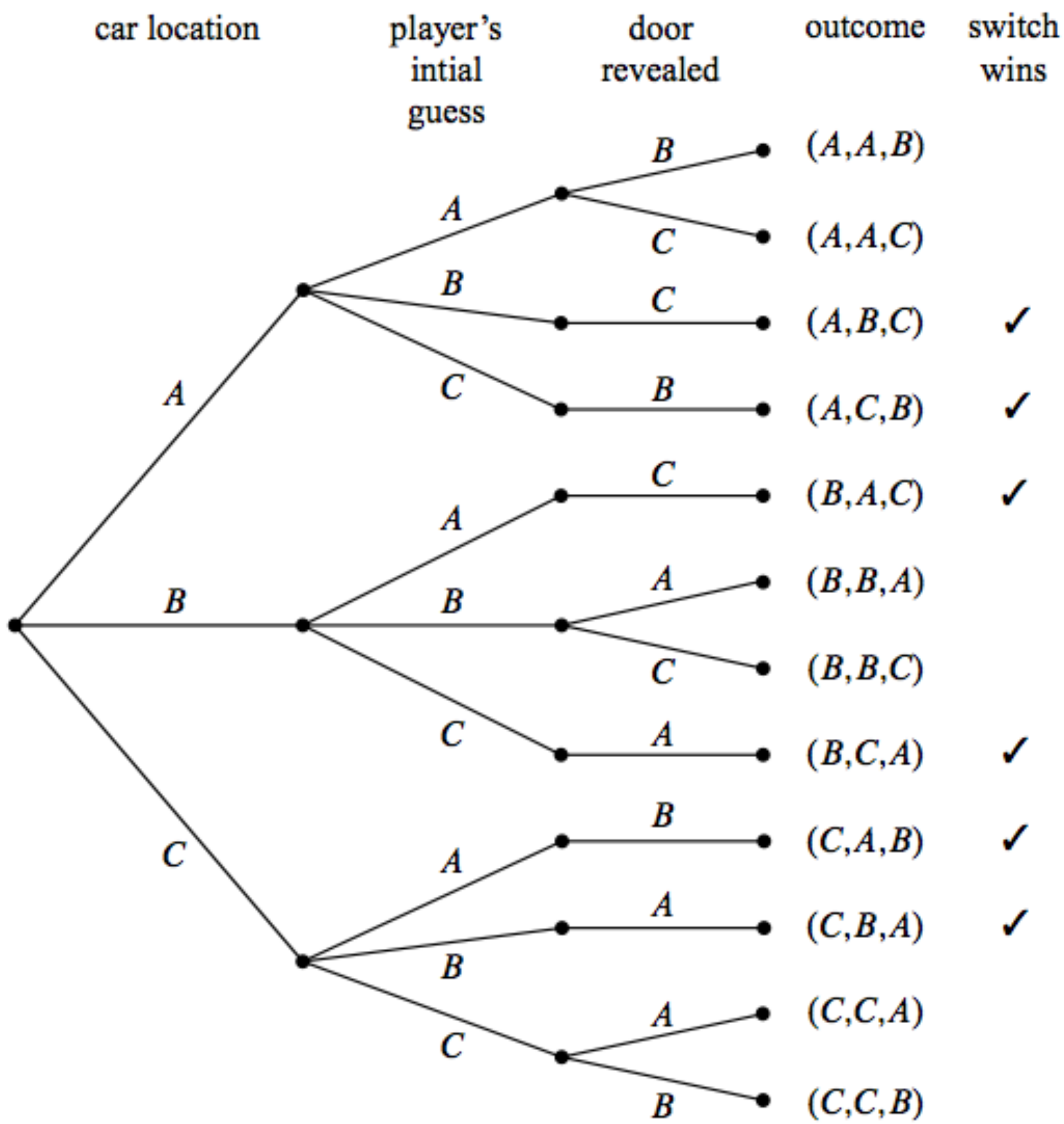
- *Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the other, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?*

- Assumptions:
  - The car is equally likely to be behind each of the doors.
  - The player is equally likely to pick each of the three doors, regardless of the car's location
  - After the player picks a door, the host ***must*** open a different door with a goat behind it and offer the player the choice of staying with the original door or switching
  - If the host has a choice of which door to open, then he is equally likely to select each of them.

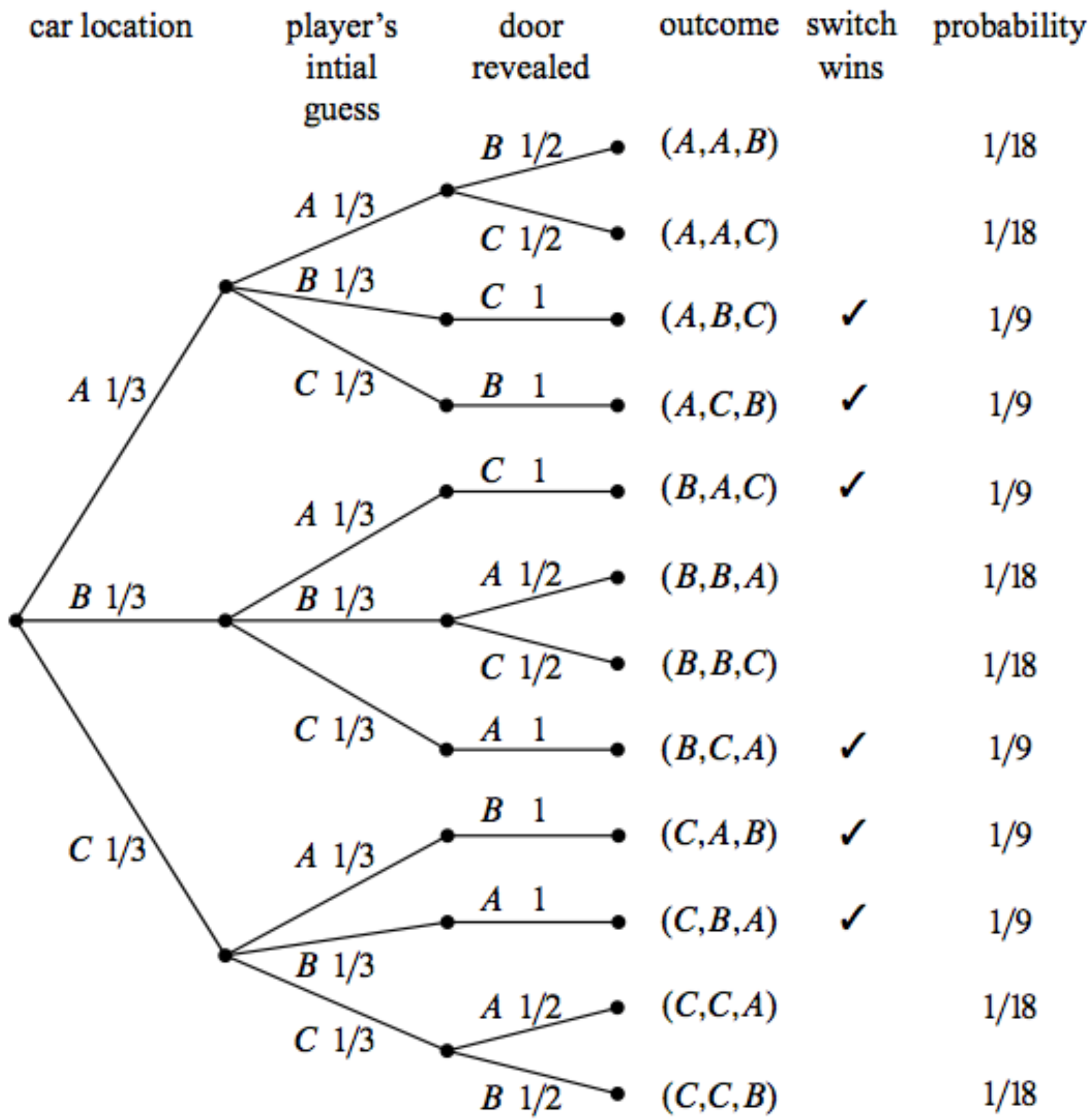
Find the Sample Space



Define events of interest



Determine outcome probabilities



Bizarre?

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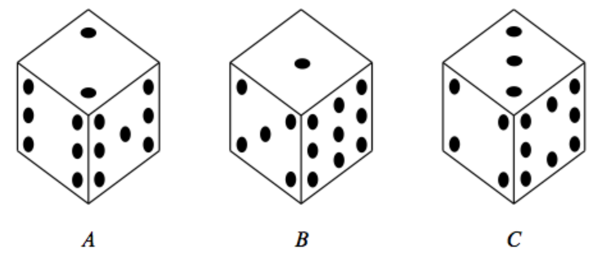
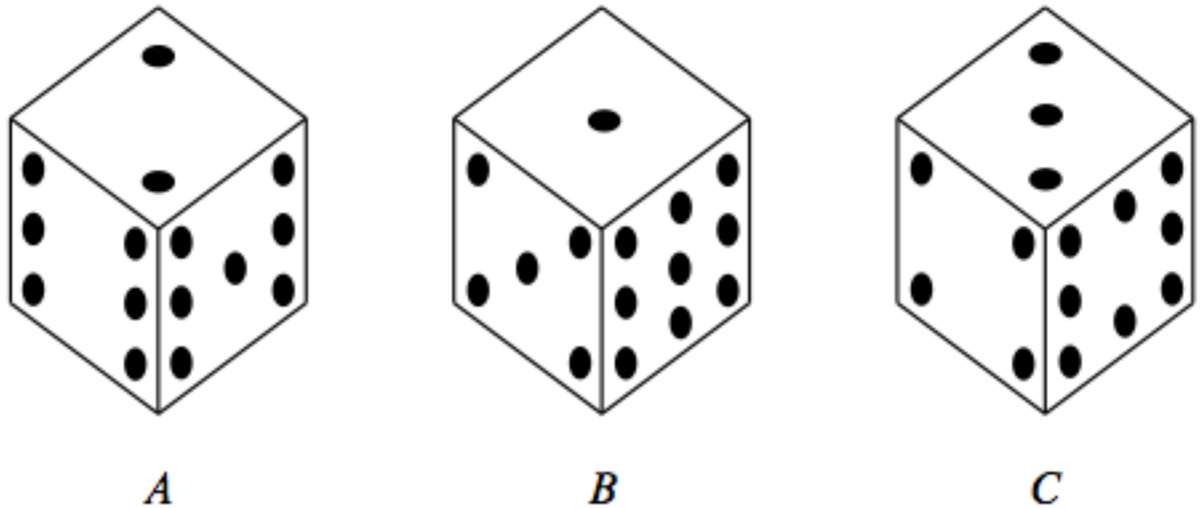


- Final Wager: $1000
  - Instead of rolling each die once, you each roll twice, and score is sum of rolls
  - This time he agrees to go first.

- He chooses B.
- So naturally, you choose A.

Conclusion



<u>One roll:</u> $A \succ B \succ C \succ A,$

<u>Two rolls:</u> $A \prec B \prec C \prec A,$