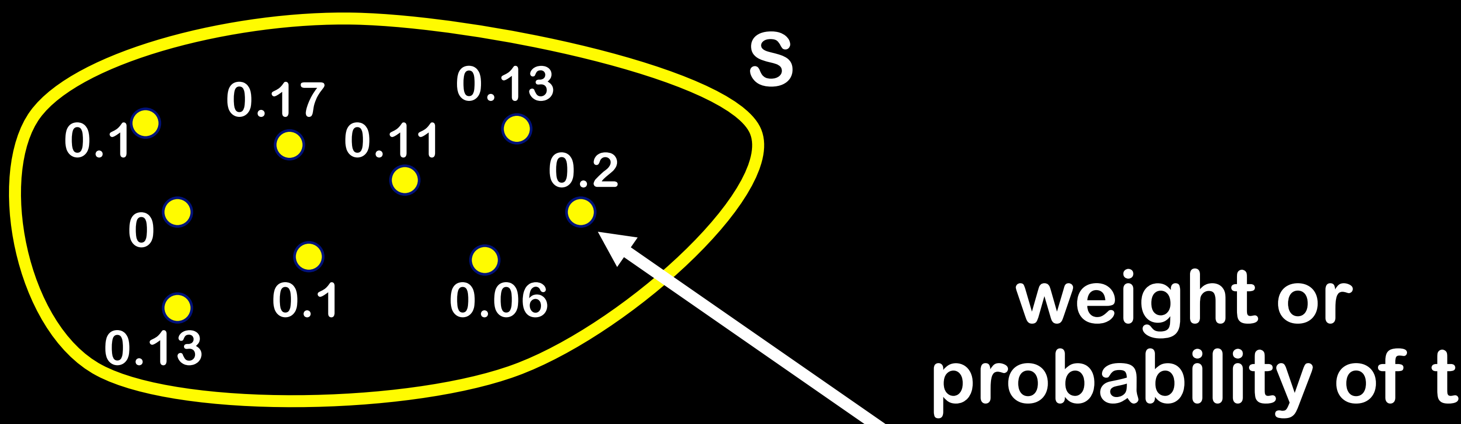
15-251

Great Theoretical Ideas in Computer Science

CMU

Sample Space



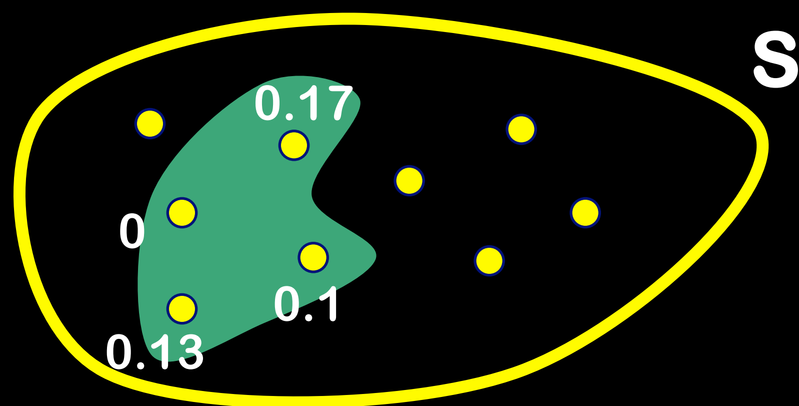
Sample space

$D(t) = p(t) = 0.2$

Events

Any set $E \subseteq S$ is called an event

$$Pr_D[E] = \sum_{t \in E} p(t)$$



$$Pr_D[E] = 0.4$$

Random Variable

Let S be sample space in a probability distribution

A Random Variable is a function from S to reals

Examples:

X = value of white die in a two-dice roll

X(3,4) = 3, X(1,6) = 1

Y = sum of values of the two dice

Y(3,4) = 7, Y(1,6) = 7

Notational Conventions

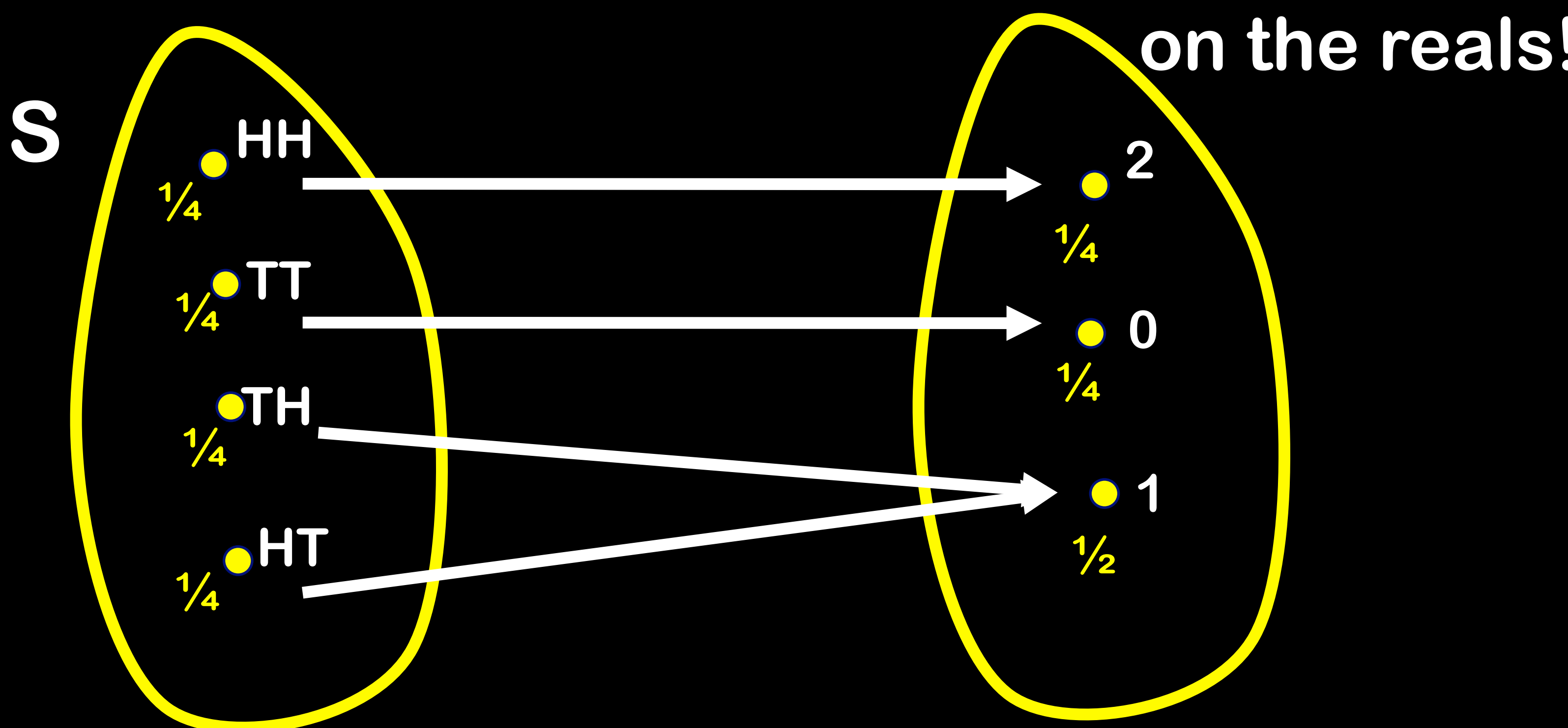
Use letters like **A, B, C** for events

Use letters like **X, Y, f, g** for R.V.'s

R.V. = random variable

Two Coins Tossed

X: {TT, TH, HT, HH} → {0, 1, 2} counts the number of heads



Two Views of Random Variables

Input to the function is random

Think of a R.V. as

A function from S to the reals R

Or think of the induced distribution on R

Randomness is "pushed" to the values of the function

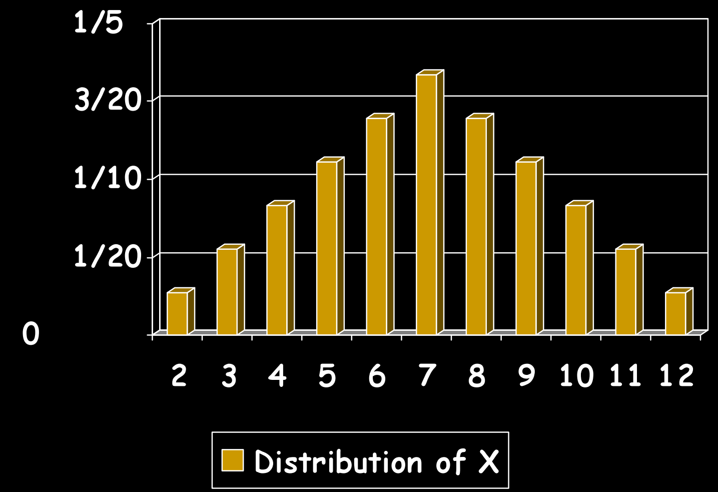
Given a distribution, a random variable transforms it into a distribution on reals

Two dice

I throw a white die and a black die.

Sample space S =
{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

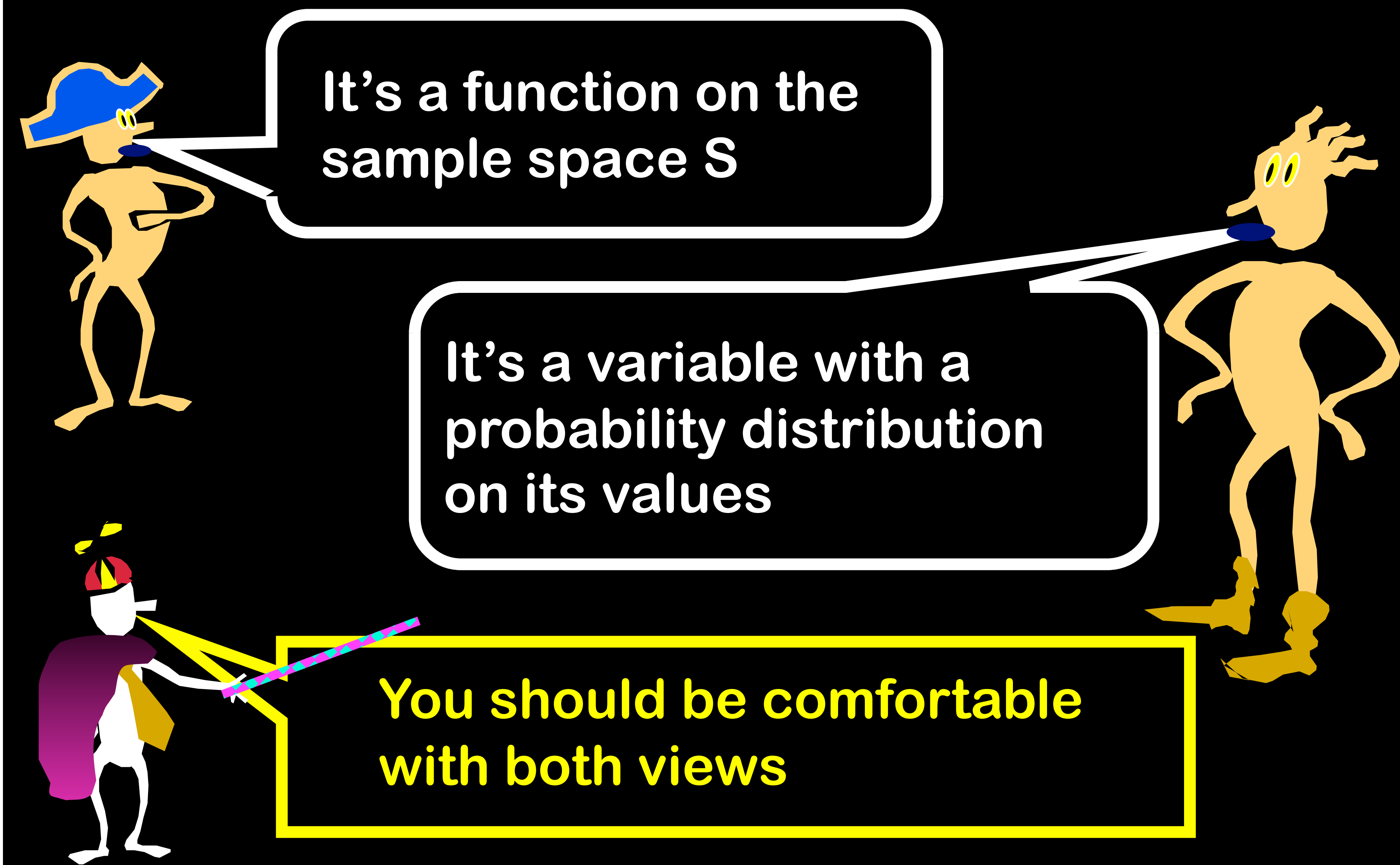
Probability mass function p.m.f.



X = sum of both dice

function with X(1,1) = 2, X(1,2) = 3, …, X(6,6)=12

It's a Floor Wax And a Dessert Topping



Two dice

I throw a white die and a black die.

Sample space S =
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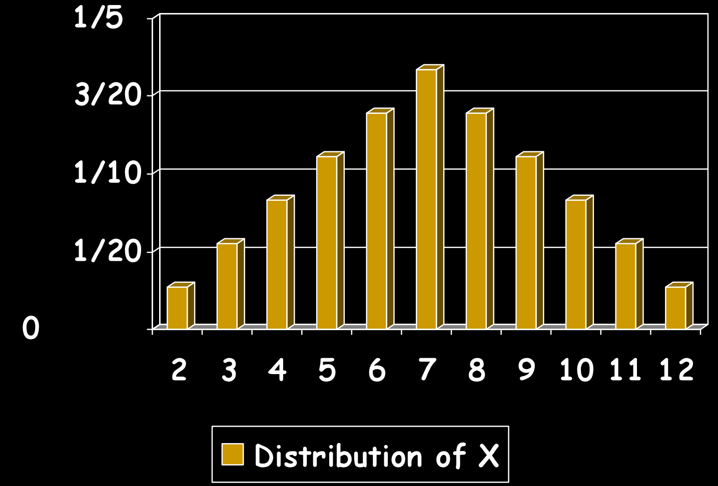
Probability mass function p.m.f.



X = sum of both dice

function with X(1,1) = 2, X(1,2) = 3, …, X(6,6)=12

From Random Variables to Events

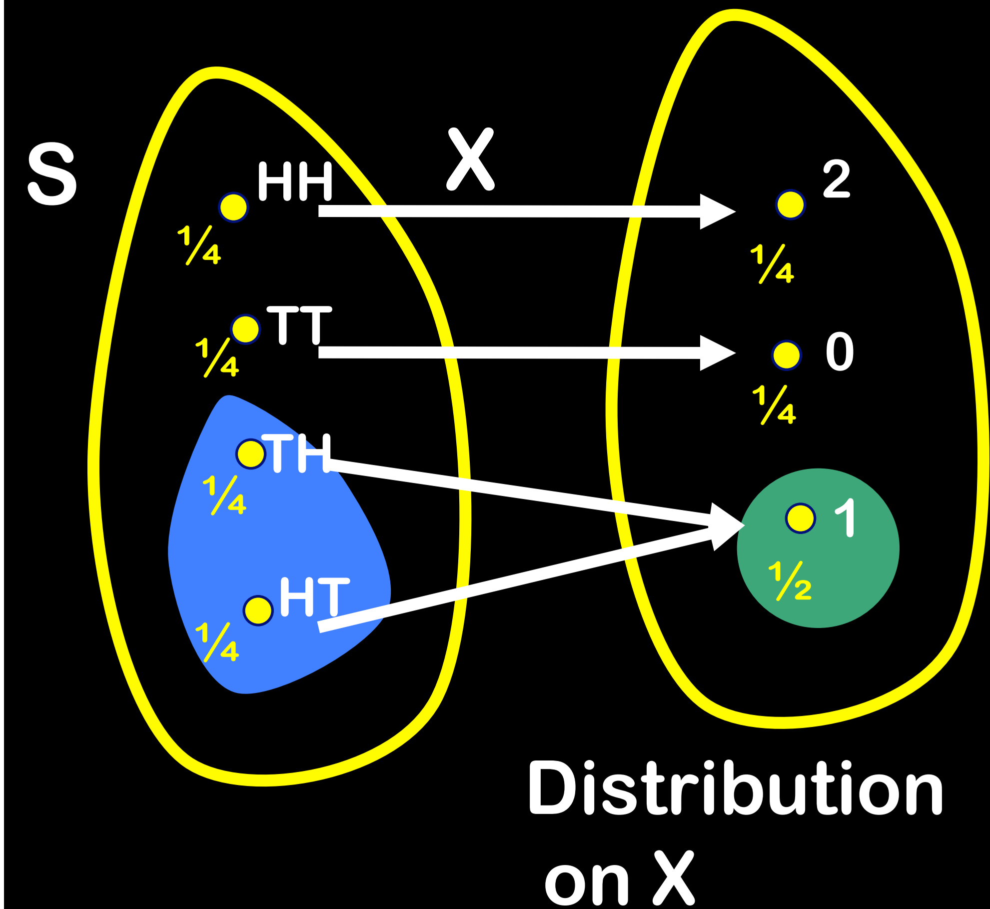
Note that each event in the induced distribution corresponds to some event in the original one.

For any random variable X and value a, we can define the event A that X = a

$$\Pr(A) = \Pr(X=a) = \Pr(\{t \in S \mid X(t)=a\})$$

Two Coins Tossed

X: {TT, TH, HT, HH} → {0, 1, 2} counts # of heads



Pr(X = a) =
Pr({t ∈ S| X(t) = a})

Pr(X = 1)

= Pr({t ∈ S| X(t) = 1})

= Pr({TH, HT}) = ½

Two dice

I throw a white die and a black die.

Sample space S =

{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
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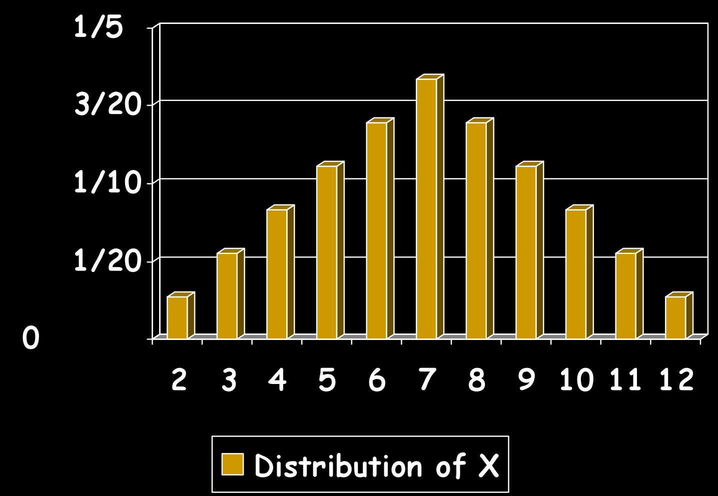


X = sum of both dice

Pr(X = 7) = 6/36 = 1/6

Definition: Expectation

The expectation, or expected value of a random variable X is written as E[X], and is

$$E[X] = \sum_{t \in S} \Pr(t)\, X(t) = \sum_{k} k \Pr[X = k]$$

X is a function on the sample space S

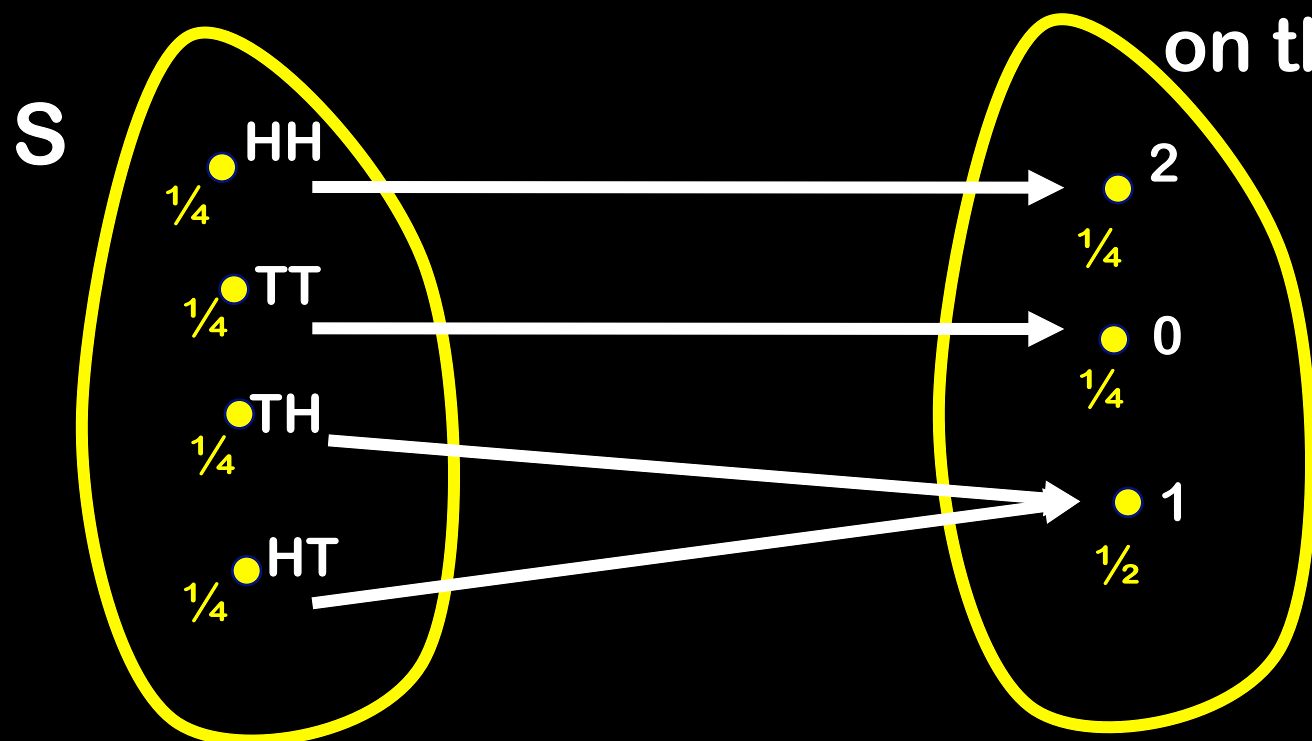
X has a prob. distribution on its values

(assuming X takes values in the naturals)

X = # of heads

$$E[X] = \sum_{t \in S} Pr(t)\, X(t) = \sum_{k} k\, Pr[X = k]$$

Distribution on the reals!



Quick check:

$$\sum_{t \in S} \Pr(t)\, X(t) = \sum_{k} k \Pr[X = k]$$

$\Pr(X = a) = \Pr(\{t \in S \mid X(t) = a\})$

A Quick Calculation…

What if I flip a coin 3 times? What is the expected number of heads?

$E[X] = (1/8)\times 0 + (3/8)\times 1 + (3/8)\times 2 + (1/8)\times 3 = 1.5$

But $Pr[\, X = 1.5 \,] = 0$

Moral: don't always expect the expected.
$Pr[\, X = E[X] \,]$ may be 0 !

Type Checking

P(B)   B must be an event

E(X)   X must be a R.V.

cannot do P(R.V.) or E(event)

Operations on R.V.s

You can sum them, take differences,
or do most other math operations…

E.g., $(X + Y)(t) = X(t) + Y(t)$

$(X*Y)(t) = X(t) * Y(t)$

$(X^Y)(t) = X(t)^{Y(t)}$

Random variables and expectations allow us to give elegant solutions to problems that seem really really messy…

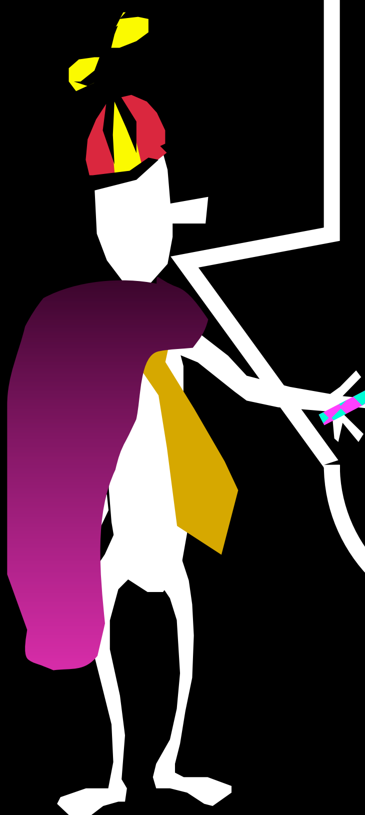
If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?

On average, in class of size m, how many pairs of people will have the same birthday?

Pretty messy with direct counting…

The new tool is called
"Linearity of Expectation"

Linearity of Expectation



If $Z = X+Y$, then

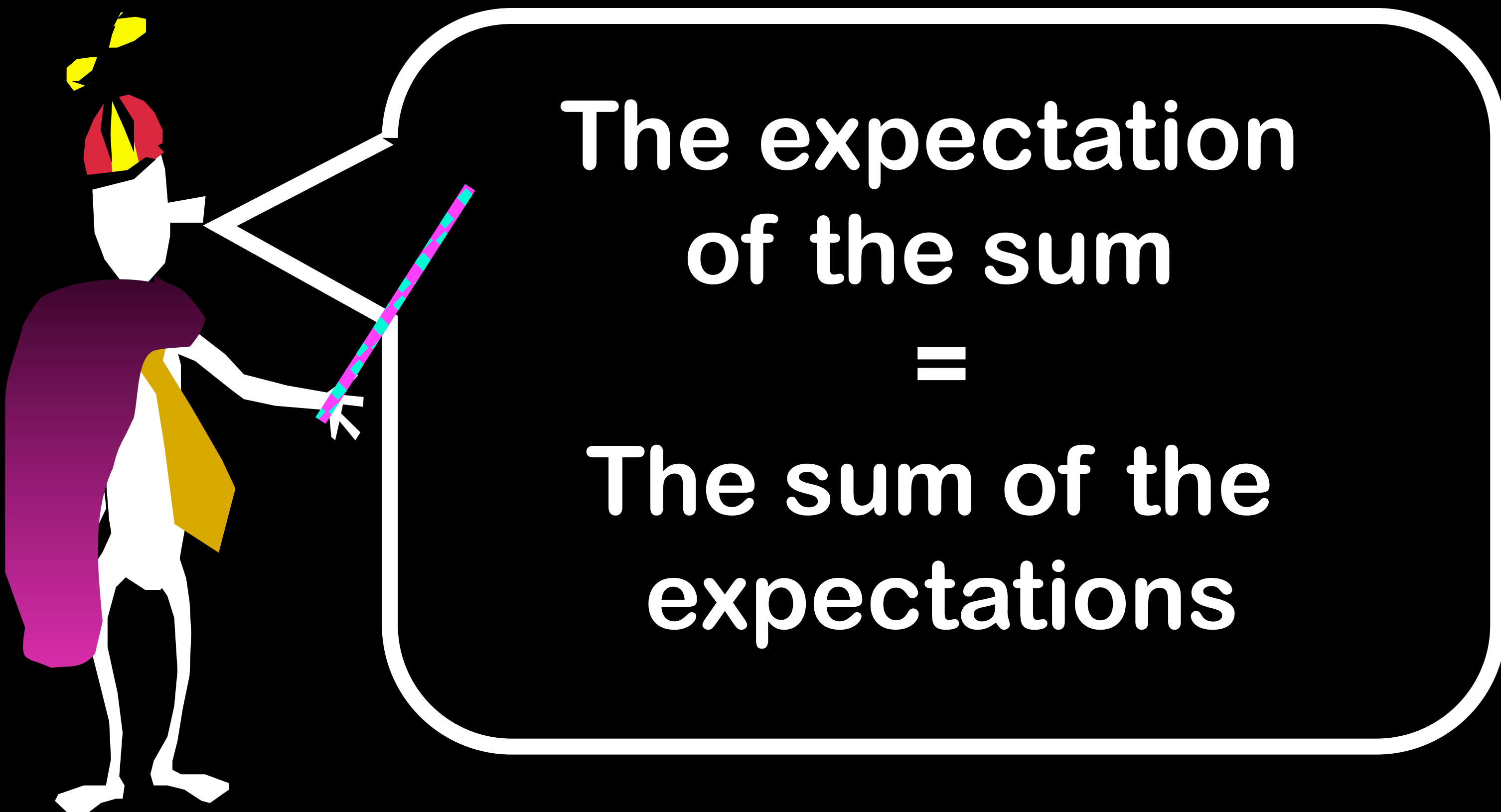
$$E[Z] = E[X] + E[Y]$$

Even if X and Y are not independent

By Induction

$$E[X_1 + X_2 + \ldots + X_n] =$$

$$E[X_1] + E[X_2] + \ldots. + E[X_n]$$



Expectation of a Sum of r.v.s

= Sum of their Expectations

even when r.v.s are not independent!

Expectation of a Product of r.v.s

= Product of their Expectations

ONLY when r.v.s are independent!

Independence for r.v.s

Two random variables X and Y are independent if for every a,b, the events X=a and Y=b are independent

How about the case of X=1st die, Y=2nd die?

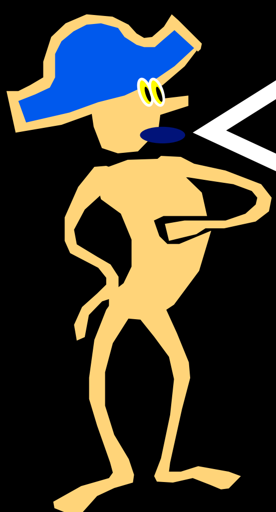
Let’s test our
Linearity of Expectation
chops…

If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?

Hmm…

$\sum_k$ k Pr(exactly k letters end up in correct envelopes)

= $\sum_k$ k (…aargh!!…)

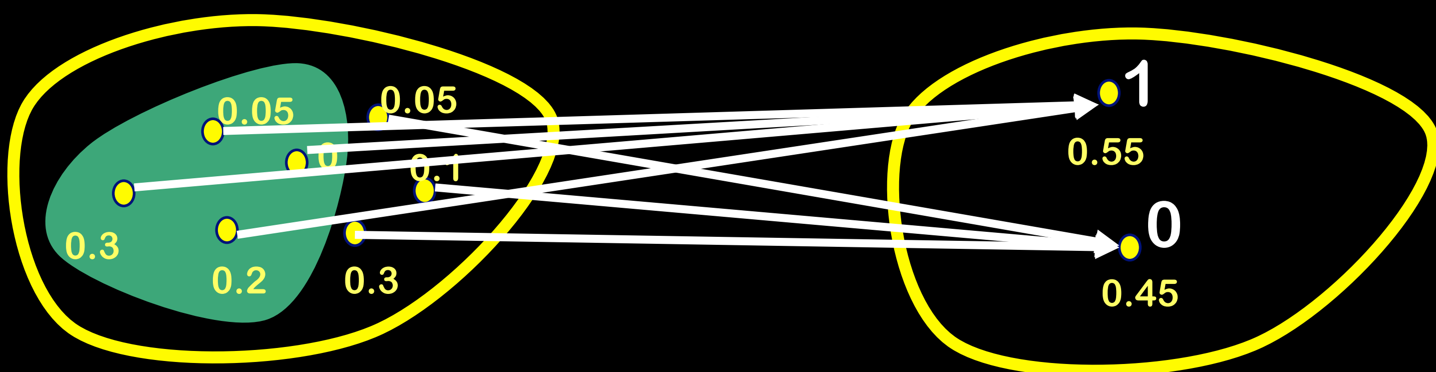


Indicator Random Variables

For any event **A**, can define the "indicator random variable" for event **A**:

$$X_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases}$$

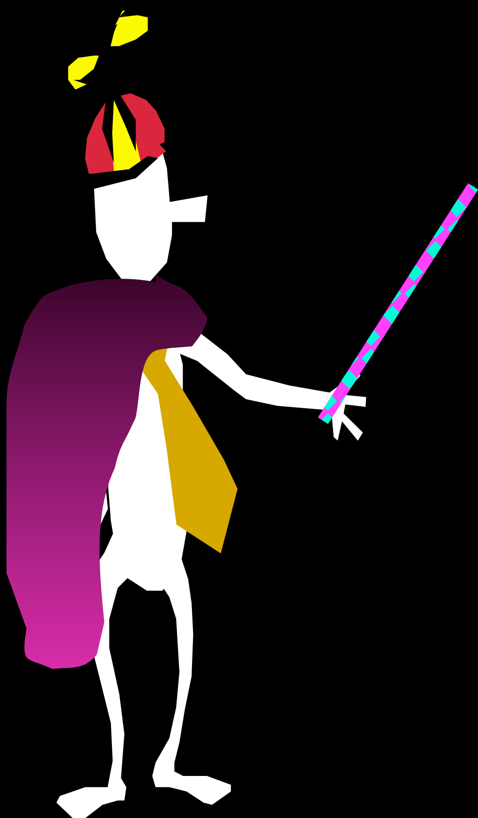
$$\boxed{E[X_A] = 1 \times \Pr(X_A = 1) = \Pr(A)}$$



Use Linearity of Expectation

Let A_i be the event the i^{th} letter ends up in its correct envelope

Let X_i be the "indicator" R.V. for A_i



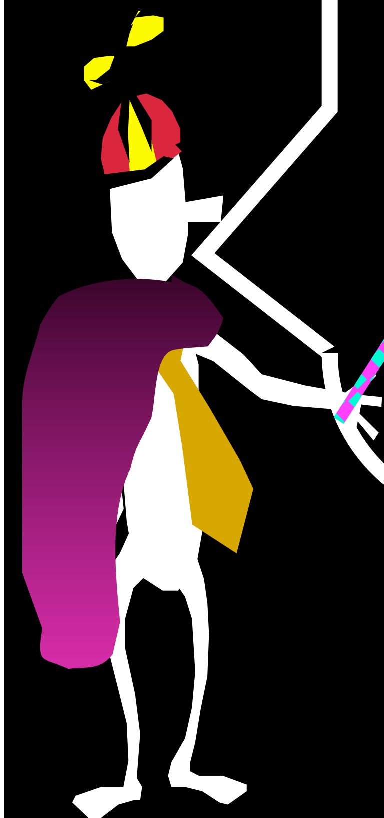
$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = X_1 + \ldots + X_{100}$

We are asking for **E[Z]**

$E[X_i] = \Pr(A_i) = 1/100$

So $E[Z] = 1$



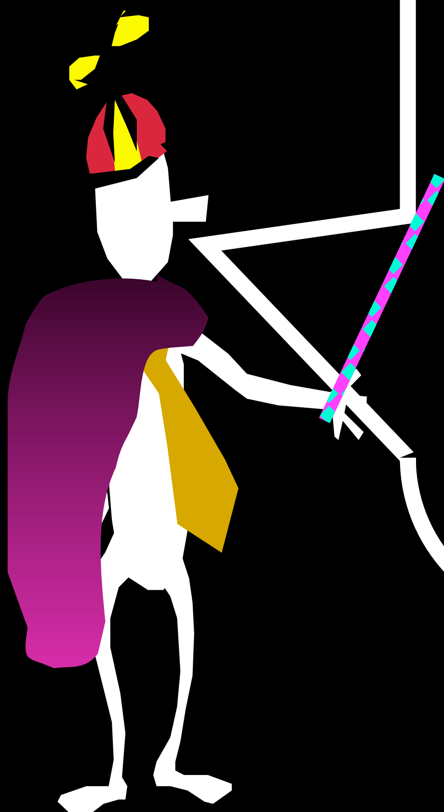
So, in expectation, 1 letter will be in the same correct envelope

Pretty neat: it doesn't depend on how many letters!

Question: were the X_i independent?

No! E.g., think of n=2

Use Linearity of Expectation



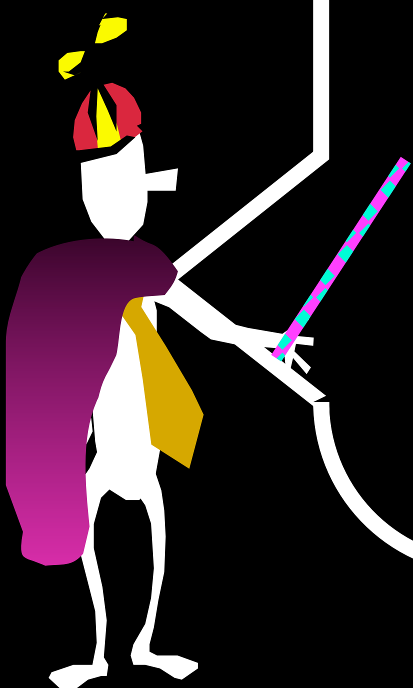
General approach:

View thing you care about as expected value of some RV

Write this RV as sum of simpler RVs

Solve for their expectations and add them up!

Example #2



We flip n coins of bias p. What is the expected number of heads?

We could do this by summing

$$\sum_k k \Pr(X = k) = \sum_k k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= n.p$$

But now we know a better way!

Linearity of Expectation!

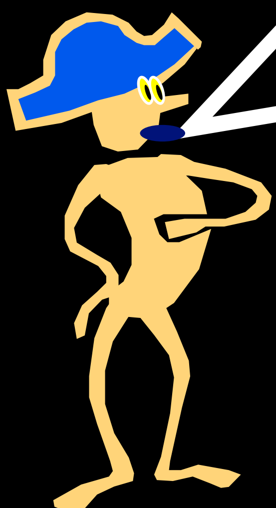
Let X = number of heads when n independent coins of bias p are flipped

Break X into n simpler RVs:

$$X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ coin is heads} \\ 0 & \text{if the } i^{th} \text{ coin is tails} \end{cases}$$

$E[X] = E[\Sigma_i X_i] = \Sigma_i E[X_i] = \Sigma_i p =$ **np**

On average, in class of size m, how many pairs of people will have the same birthday?



$\sum_k$ k Pr(exactly k collisions)

= $\sum_k$ k (...aargh!!!!...)

Use linearity of expectation

Suppose we have m people each with a uniformly chosen birthday from 1 to 365

X = number of pairs of people with the same birthday

E[X] = ?

X = number of pairs of people with the same birthday

$E[X] = ?$

Use $m(m-1)/2$ indicator variables, one for each pair of people

$X_{jk} = 1$ if person j and person k have the same birthday; else 0

$$E[X_{jk}] = (1/365)\ 1 + (1 - 1/365)\ 0$$
$$= 1/365$$

X = number of pairs of people with the same birthday

X_{jk} = 1 if person j and person k have the same birthday; else 0

$$E[X_{jk}] = (1/365)\,1 + (1 - 1/365)\,0 = 1/365$$

$$E[X] = E\left[\sum_{1 \le j < k \le m} X_{jk}\right] = \sum_{1 \le j < k \le m} E[X_{jk}] = m(m-1)/2 \times 1/365$$

E.g., setting m = 23, we get

E[# pairs with same birthday]

= 0.691…

How does this compare to

Pr[at least one pair has same birthday] ?