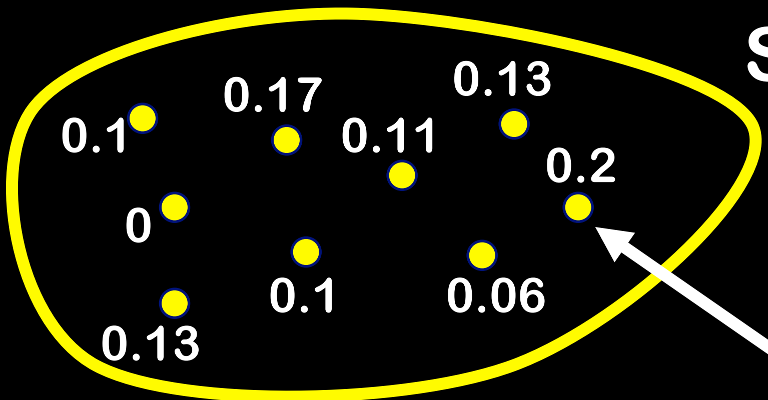
15-251

Great Theoretical Ideas in Computer Science

CMU

Sample Space



Sample space

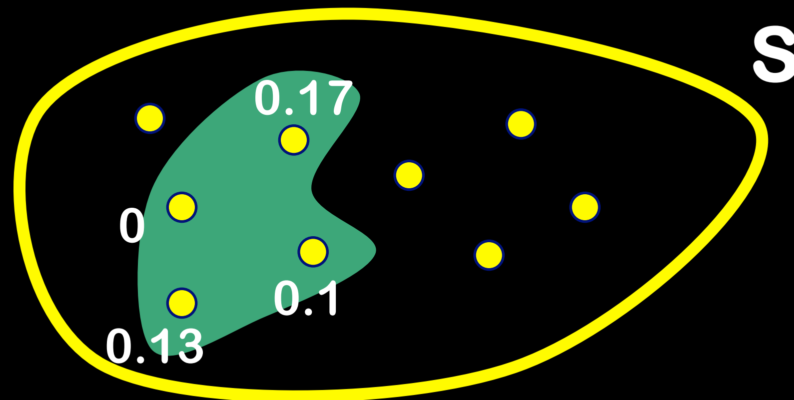
weight or probability of t

D(t) = p(t) = 0.2

Events

Any set $E \subseteq S$ is called an event

$$\Pr_D[E] = \sum_{t \in E} p(t)$$



$$\Pr_D[E] = 0.4$$

Random Variable

Let S be sample space in a probability distribution

A Random Variable is a function from S to reals

Examples:

X = value of white die in a two-dice roll

X(3,4) = 3, X(1,6) = 1

Y = sum of values of the two dice

Y(3,4) = 7, Y(1,6) = 7

Notational Conventions

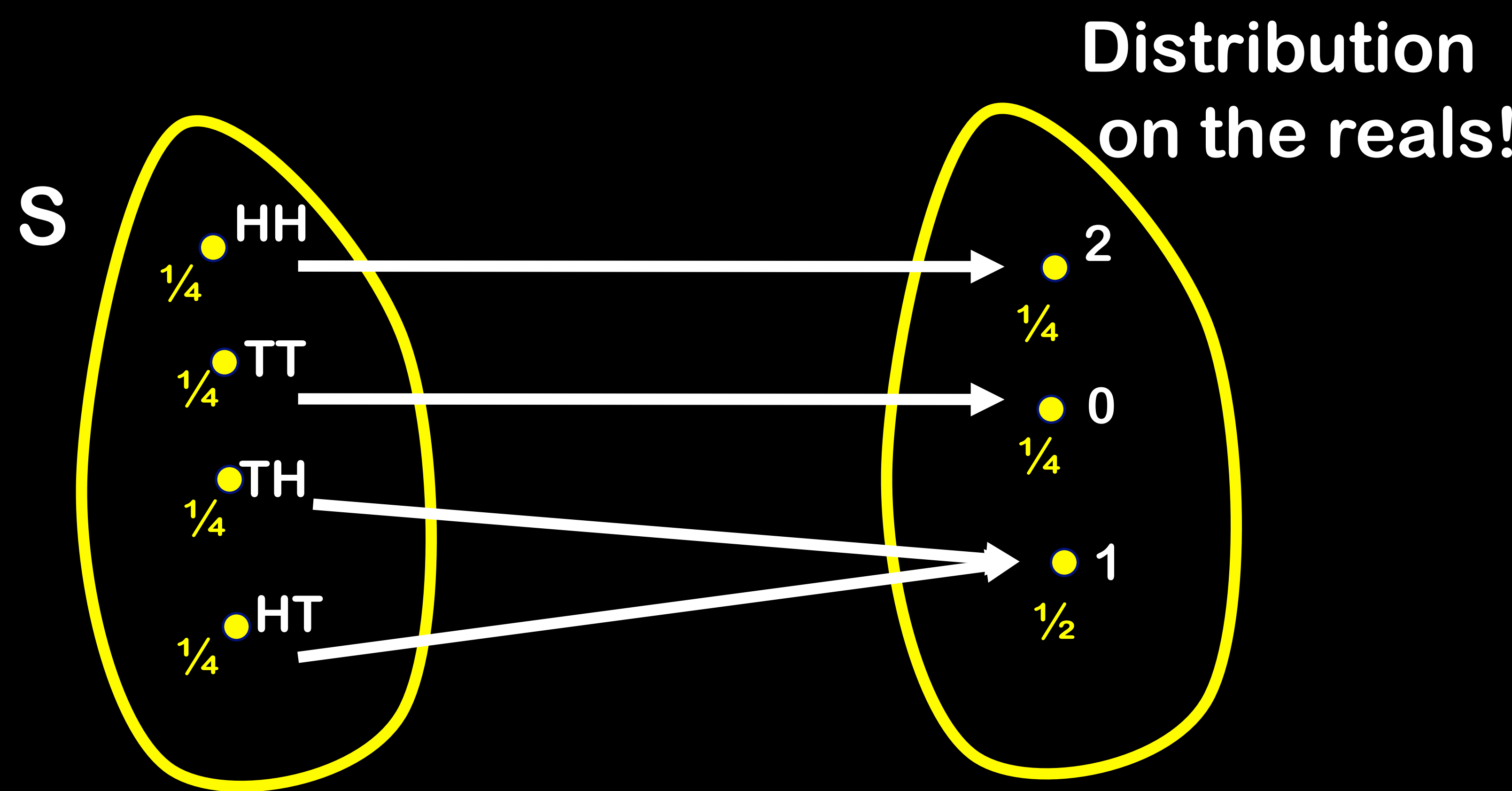
Use letters like A, B, C for events

Use letters like X, Y, f, g for R.V.'s

R.V. = random variable

Two Coins Tossed

X: {TT, TH, HT, HH} → {0, 1, 2} counts the number of heads



Two Views of Random Variables

Input to the function is random

Think of a R.V. as

A function from S to the reals R

Or think of the induced distribution on R

Randomness is "pushed" to the values of the function

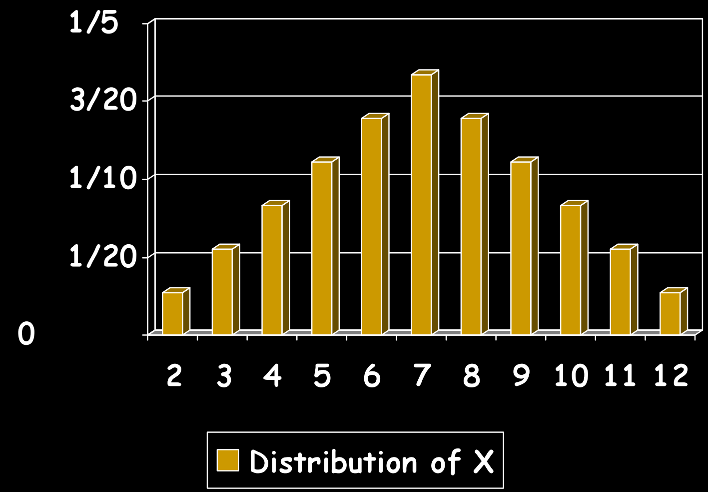
Given a distribution, a random variable transforms it into a distribution on reals

Two dice

I throw a white die and a black die.

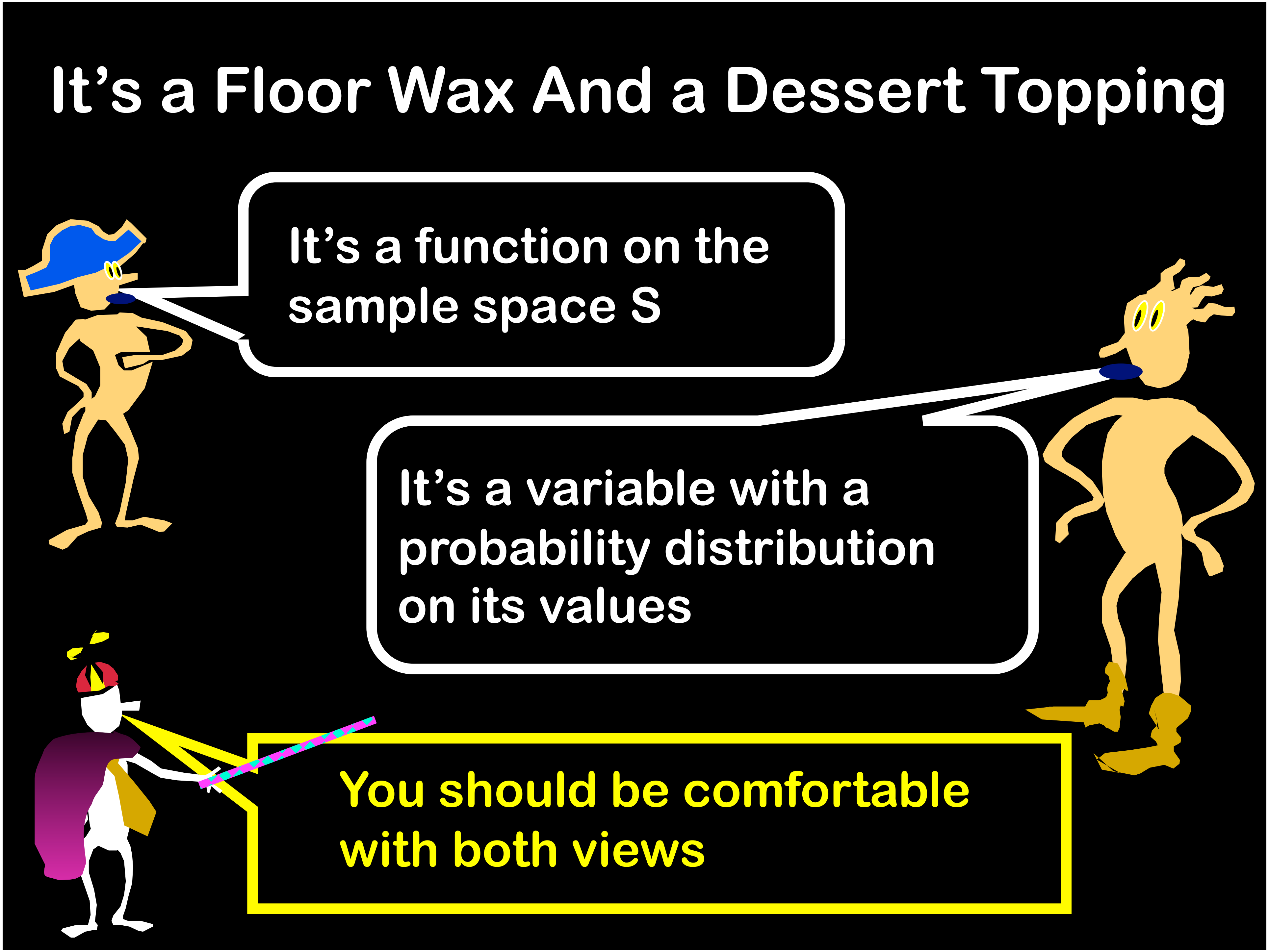
Sample space S =
{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

Probability mass function p.m.f.



X = sum of both dice

function with X(1,1) = 2, X(1,2) = 3, …, X(6,6)=12

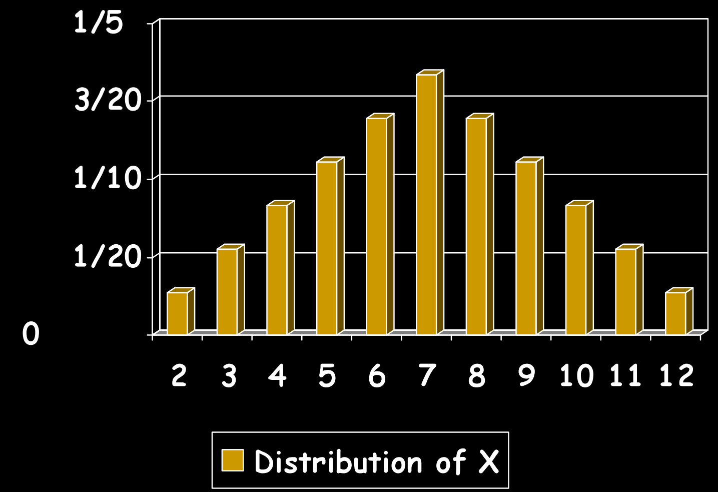

It's a Floor Wax And a Dessert Topping
It's a function on the sample space S
It's a variable with a probability distribution on its values
You should be comfortable with both views

Two dice

I throw a white die and a black die.

Sample space S =
{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

Probability mass function p.m.f.



X = sum of both dice

function with X(1,1) = 2, X(1,2) = 3, …, X(6,6)=12

From Random Variables to Events

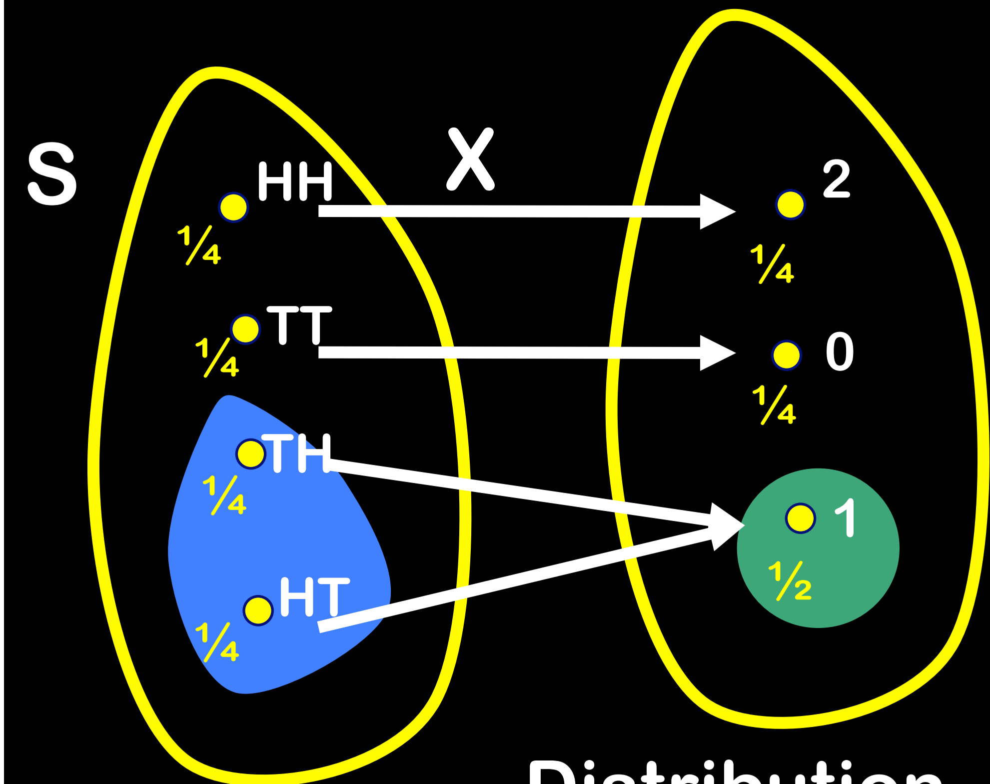
Note that each event in the induced distribution corresponds to some event in the original one.

For any random variable **X** and value **a**, we can define the event **A** that **X = a**

$$\Pr(A) = \Pr(X=a) = \Pr(\{t \in S \mid X(t)=a\})$$

Two Coins Tossed

X: {TT, TH, HT, HH} → {0, 1, 2} counts # of heads



Distribution on X

$Pr(X = a) = Pr(\{t \in S | X(t) = a\})$

$Pr(X = 1)$

$= Pr(\{t \in S | X(t) = 1\})$

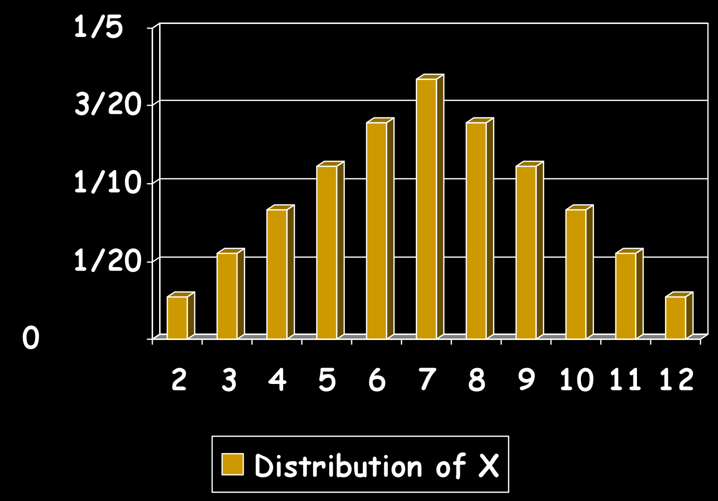
$= Pr(\{TH, HT\}) = ½$

Two dice

I throw a white die and a black die.

Sample space S =

{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }



X = sum of both dice

Pr(X = 7) = 6/36 = 1/6

Definition: Expectation

The expectation, or expected value of a random variable X is written as E[X], and is

$$E[X] = \sum_{t \in S} \Pr(t)\, X(t) = \sum_{k} k \Pr[X = k]$$

X is a function on the sample space S

X has a prob. distribution on its values

(assuming X takes values in the naturals)

X = # of heads

$$E[X] = \sum_{t \in S} \Pr(t)\, X(t) = \sum_{k} k \Pr[X = k]$$