independence

Defn: Two events E and F are *independent* if

$P(EF) = P(E)\,P(F)$

If $P(F)>0$, this is equivalent to: $P(E|F) = P(E)$ (proof below)

Otherwise, they are called *dependent*

Roll two dice, yielding values D_1 and D_2

1) $E = \{ D_1 = 1 \}$

$F = \{ D_2 = 1 \}$

$P(E) = 1/6, \;\; P(F) = 1/6, \;\; P(EF) = 1/36$

$P(EF) = P(E) \bullet P(F) \Rightarrow$ E and F *independent*

Intuitive; the two dice are not physically coupled

2) $G = \{D_1 + D_2 = 5\} = \{(1,4),(2,3),(3,2),(4,1)\}$

$P(E) = 1/6$, $P(G) = 4/36 = 1/9$, $P(EG) = 1/36$

not independent!

E, G are *dependent* events

The dice are still not physically coupled, but "$D_1 + D_2 = 5$" couples them <u>mathematically:</u> info about D_1 constrains D_2. (But dependence/ independence not always intuitively obvious; "use the definition, Luke".)

Two events E and F are *independent* if

$P(EF) = P(E)\,P(F)$

If $P(F)>0$, this is equivalent to: $P(E|F) = P(E)$

Otherwise, they are called *dependent*

Three events E, F, G are independent if

$$\begin{aligned} P(EF) &= P(E)\,P(F) \\ P(EG) &= P(E)\,P(G) \qquad \textit{and} \qquad P(EFG) = P(E)\,P(F)\,P(G) \\ P(FG) &= P(F)\,P(G) \end{aligned}$$

Example: Let X, Y be each $\{-1,1\}$ all outcomes equally likely

$E = \{X = 1\}$, $F = \{Y = 1\}$, $G = \{XY = 1\}$

$P(EF) = P(E)P(F)$, $P(EG) = P(E)P(G)$, $P(FG) = P(F)P(G)$

but $P(EFG) = 1/4$!!

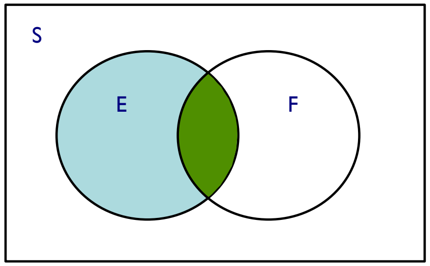
In general, events $E_1, E_2, \ldots, E_n$ are independent if for *every subset* S of {1,2,…, n}, we have

$$P\left(\bigcap_{i \in S} E_i\right) = \prod_{i \in S} P(E_i)$$

(Sometimes this property holds only for small subsets S. E.g., E, F, G on the previous slide are *pairwise* independent, but not fully independent.)

Theorem: E, F independent $\Rightarrow$ E, F^c independent

$E = EF \cup EF^c$



Proof: $P(EF^c) = P(E) - P(EF)$

$= P(E) - P(E)\,P(F)$

$= P(E)\,(1-P(F))$

$= P(E)\,P(F^c)$

Theorem: $P(E)>0$, $P(F)>0$

E, F independent $\Leftrightarrow P(E|F)=P(E) \Leftrightarrow P(F|E) = P(F)$

Proof: Note $P(EF) = P(E|F)\,P(F)$, regardless of in/dep.

Assume independent. Then

$$P(E)P(F) = P(EF) = P(E|F)\,P(F) \Rightarrow P(E|F)=P(E) \quad (\div \text{ by } P(F))$$

Conversely, $P(E|F)=P(E) \Rightarrow P(E)P(F) = P(EF)$ $(\times \text{ by } P(F))$

Suppose a biased coin comes up heads with probability p, *independent* of other flips

P(n heads in n flips) = p^n

P(n tails in n flips) = $(1-p)^n$

P(exactly k heads in n flips) $= \binom{n}{k} p^k (1-p)^{n-k}$

Aside: note that the probability of *some* number of heads = $\sum_k \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1$

as it should, by the binomial theorem.

Suppose a biased coin comes up heads with probability p, *independent* of other flips

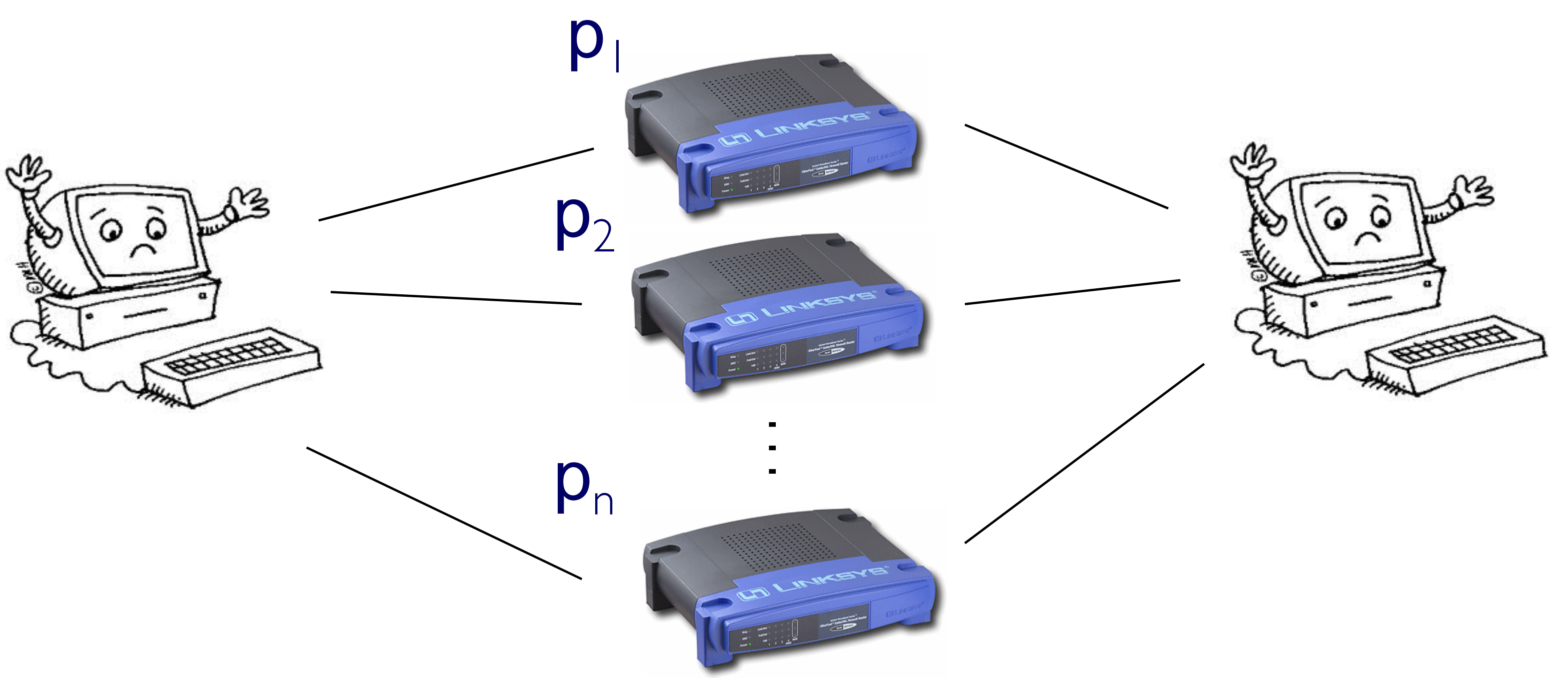
P(exactly k heads in n flips) $= \binom{n}{k} p^k (1-p)^{n-k}$

Note when p=1/2, this is the same result we would have gotten by considering n flips in the "equally likely outcomes" scenario. But $p \neq 1/2$ makes that inapplicable. Instead, the *independence* assumption allows us to conveniently assign a probability to each of the 2^n outcomes, e.g.:

Pr(HHTHTTT) = $p^2(1-p)p(1-p)^3 = p^{\#H}(1-p)^{\#T}$

network failure

Consider the following parallel network



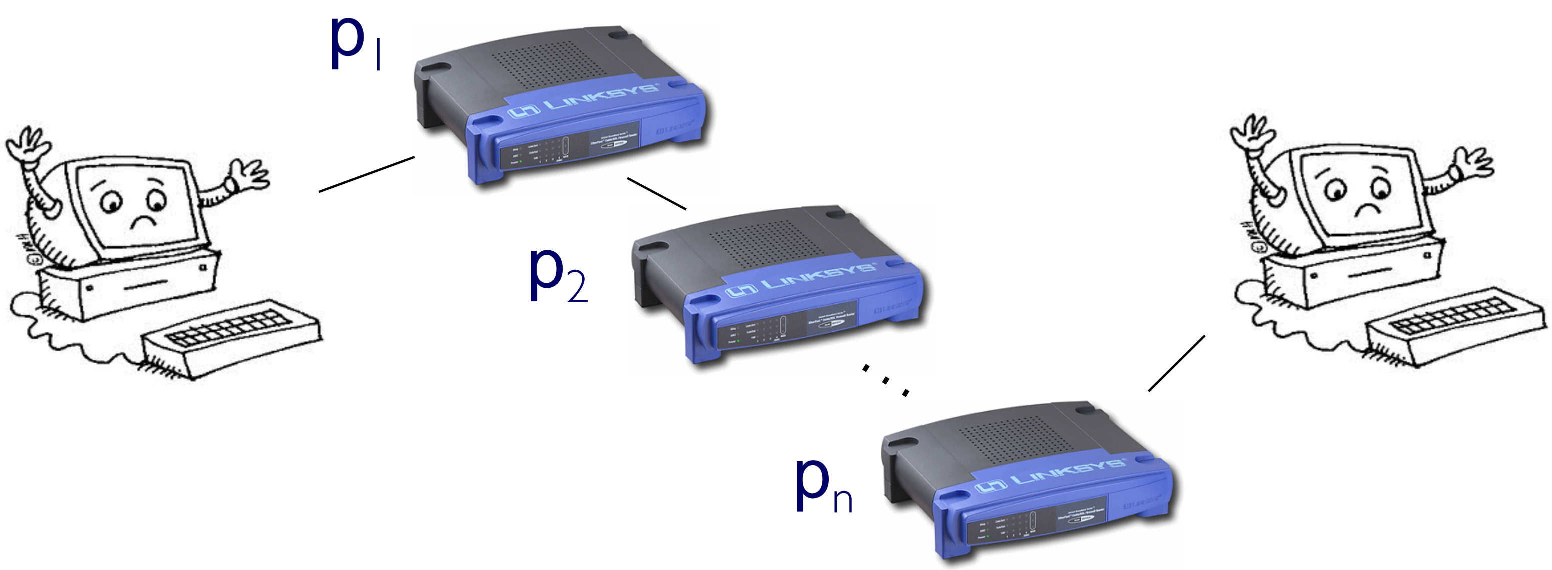
n routers, i^{th} has probability p_i of failing, independently

P(there is functional path) = 1 – P(all routers fail)

$$= 1 - p_1 p_2 \cdots p_n$$

network failure

Contrast: a *series* network



n routers, i^{th} has probability p_i of failing, independently

P(there is functional path) =

P(*no* routers fail)

$$= (1 - p_1)(1 - p_2) \cdots (1 - p_n)$$

Recall: Two events E and F are independent if

$P(EF) = P(E)\,P(F)$

If E & F are independent, does that tell us anything about

$P(EF|G), P(E|G), P(F|G),$

when G is an arbitrary event? In particular, is

$P(EF|G) = P(E|G)\,P(F|G)$?

In general, *no*.

Roll two 6-sided dice, yielding values D_1 and D_2

$E = \{ D_1 = 1 \}$

$F = \{ D_2 = 6 \}$

$G = \{ D_1 + D_2 = 7 \}$

E and F are independent

$P(E|G) = 1/6$

$P(F|G) = 1/6$, but

$P(EF|G) = 1/6$, *not* $1/36$

so E|G and F|G are not independent!

Definition:

Two events E and F are called *conditionally independent given G*, if

$P(EF|G) = P(E|G)\ P(F|G)$

Or, equivalently (assuming $P(F)>0$, $P(G)>0$),

$P(E|FG) = P(E|G)$

conditioning can also break DEPENDENCE

Randomly choose a day of the week

A = { It is not a Monday }

B = { It is a Saturday }

C = { It is the weekend }

A and B are dependent events

$P(A) = 6/7$, $P(B) = 1/7$, $P(AB) = 1/7$.

Now condition both A and B on C:

$P(A|C) = 1$, $P(B|C) = ½$, $P(AB|C) = ½$

$P(AB|C) = P(A|C)\, P(B|C) \Rightarrow A|C$ and $B|C$ independent

Dependent events can become independent by conditioning on additional information!

Another reason why conditioning is so useful

Events E & F are *independent* if

P(EF) = P(E) P(F), or, equivalently P(E|F) = P(E) (if p(E)>0)

More than 2 events are indp if, for *all subsets,* joint probability = product of separate event probabilities

Independence can greatly simplify calculations

For fixed G, conditioning on G gives a probability measure, P(E|G)

But "conditioning" and "independence" are orthogonal:

Events E & F that are (unconditionally) independent may become dependent when conditioned on G

Events that are (unconditionally) dependent may become independent when conditioned on G