Expected value of random variable X given event A

$$E(X|A) = \sum_{x \in Range(X)} x Pr(X = x|A)$$

Law of Total Expectation (example)

49.8% of population male

Average height 5'11" (men) 5'5" (female)

$$E(H) = E(H|M)Pr(M) + E(H|F)Pr(F)$$
$$= 5\frac{11}{12} \cdot 0.498 + 5\frac{5}{12} \cdot 0.502$$

X random variable on a sample space S

$A_1, A_2, \ldots, A_k$ partition of S

$$
\begin{aligned}
E(X) &= \sum_i E(X|A_i)Pr(A_i) \\
&= \sum_i \sum_x xPr(X = x|A_i)Pr(A_i) \\
&= \sum_x \sum_i xPr(X = x|A_i)Pr(A_i) \\
&= \sum_x x \sum_i Pr(X = x|A_i)Pr(A_i) \\
&= \sum_i xPr(X = x)
\end{aligned}
$$

Law of Total Expectation : Application

System that fails in step i independently with probability p

X # steps to fail

E(X) ?

Let A be the event that system fails in first step.

$$E(X) = E(X|A)Pr(A) + E(X|\overline{A})Pr(\overline{A})$$

$$= p + (1 + E(X))(1 - p)$$

$$= 1 + (1 - p)E(X)$$

$$E(X) = \frac{1}{p}$$

Law of Total Expectation : Example

A miner is trapped in a mine containing 3 doors.

- The 1st door leads to a tunnel that will take him to safety after 3 hours.
- The 2nd door leads to a tunnel that returns him to the mine after 5 hours.
- The 3rd door leads to a tunnel that returns him to the mine after 7 hours.

At all times, he is equally likely to choose any one of the doors.

E(time to reach safety) ?

Algorithms and randomized algorithms

- Binary search: Given a sorted array of n numbers, determine if the array contains the number 153.
- Given an array of unsorted numbers, sort them.
- Given an array of 0's and 1's, either ½ of each, or all 1's. Determine which.

Worst case running time: measure of work algorithm does

Given array of length n

If n = 0 or 1, halt

Otherwise, pick element p of array as “pivot”

Split array into subarrays: < p, = p, > p

Recursively sort subarray < p

Recursively sort subarray > p

Worst case number of comparisons?

What if we use a random pivot?

That makes it a randomized algorithm!