CDF $F_X(x) = Pr(X \le x)$ ↗ from 0 to 1

pdf $f_X(x) = \frac{d}{dx} F_X(x)$ nonnegative $\int_{-\infty}^{\infty} f(x)\,dx = 1$

↓

$F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz$

$Pr(X \in A) = \int_A f(z)\,dz$

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

Joint Distn's

Joint CDF: $F_{X,Y}(x,y) = Pr(X \le x, Y \le y)$

$F(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)\,dy\,dx$

($f(x,y)$: joint density fn)

$f(a,b) = \frac{\partial^2}{\partial a\,\partial b} F(a,b)$

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$Pr(a < X < a+da,\ b \le Y \le b+db) =$

$\int_{a}^{a+da} \int_{b}^{b+db} f(x,y)\,dy\,dx \approx f(a,b)\,da\,db$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

Independence & Conditional pmf

Indep r.v.'s $\equiv$ $f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \quad \forall x,y$

For y w/ $f_Y(y) > 0$ $\quad f_{X|Y=y}(x) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ conditional pdf

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{X,Y}(x,y) = f_Y(y)\, f_{X|Y=y}(x) = f_X(x)\, f_{Y|X=x}(y)$$

Law of Total Prob

$A_1, \dots, A_n$ disjoint events that form partition of sample space

$Pr(A_i > 0) \quad \forall i$

$$f_X(x) = \sum_{i=1}^{n} Pr(A_i)\, f_{X|A_i}(x) \qquad f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{Pr(A)} & x \in A \\ 0 & \text{o.w.} \end{cases}$$

$$Pr(E) = \int_{-\infty}^{\infty} Pr(E \mid X=x) f_X(x)\, dx$$

Law of Total Expectation

$$E(X) = \sum_{i=1}^{n} Pr(A_i) E(X \mid A_i) \qquad E(X \mid A) = \int_{-\infty}^{\infty} x f_{X|A}(x)\, dx$$

$$E(X) = \int_{-\infty}^{\infty} E(X \mid Y=y) f_Y(y)\, dy \qquad E(X \mid Y=y) = \int_{-\infty}^{\infty} x f_{X|Y=y}(x)\, dx$$

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

X cont r.v.

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{o.w} \end{cases}$$

What is C?

$$\int_0^2 C(4x - 2x^2)\, dx = 1 \qquad Pr(X > 1) = \int_1^2 \frac{3}{8}(4x - 2x^2)\, dx$$

$$C \cdot \frac{8}{3} \Rightarrow C = \frac{3}{8}$$

$X \sim N(5, \sigma^2)$

If $Pr(X > 9) = .2$ approx what is σ?

$Pr(X > 9) = Pr\left(\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\right) = .2$

$\iff 1 - \Phi\left(\frac{9-5}{\sigma}\right) = .2$

$\iff \Phi\left(\frac{9-5}{\sigma}\right) = .8$

look up in $N(0,1)$ table $\Rightarrow$ find out what

v gives $\Phi(v) = .8$

set $\frac{9-5}{\sigma} = v$

solve for σ

$X_1 \sim \exp(\lambda_1) \qquad X_2 \sim \exp(\lambda_2)$

$Pr(X_1 \leq X_2)$

$= \int_0^\infty Pr(X_1 \leq X_2 \mid X_1 = u) f_{X_1}(u)\, du$ use law of total prob

$$= \int_0^\infty \underbrace{\Pr(X_2 \geq u)}_{e^{-\lambda_2 u}} \underbrace{f_{X_1}(u)}_{\lambda_1 e^{-\lambda_1 u}} \, du$$

$$= \lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_2)u} \, du$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2} \underbrace{\int_0^\infty (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)u} \, du}_{=1}$$

density of $\exp(\lambda_1+\lambda_2)$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2}$$

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accidents a person has in a year; indep from year to year

is Poisson (λ) where $\lambda \sim \exp(1)$

i.e. proportion of pop w/ $\lambda < x$ is $1-e^{-x}$

E(# accidents in a year for random person)

$$= \int_0^\infty \underbrace{E(\#\text{accidents} \mid \lambda)}_{\text{Exp of Poisson}(\lambda)} f(\lambda) \, d\lambda$$

$$= \int_0^\infty \lambda e^{-\lambda} \, d\lambda = 1$$

Pr(random person has 0 accidents)

$$= \int_0^\infty \Pr(\text{no accidents} \mid \lambda)\, f(\lambda)\, d\lambda$$

$$= \int_0^\infty e^{-\lambda} e^{-\lambda}\, d\lambda = \int_0^\infty e^{-2\lambda}\, d\lambda = \frac{1}{2}$$

Pr(random person has 0 accidents this year | no accidents last year)

A = random person has 0 accidents this year

B = no accidents last year

$$= \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(A \cap B) = \int_0^\infty \Pr(A \cap B \mid \lambda)\, f(\lambda)\, d\lambda$$

$$= \int_0^\infty e^{-2\lambda} e^{-\lambda}\, d\lambda = \frac{1}{3}$$

$$= \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Recipe for finding pdf of g(X)

$Y = g(X)$

1) Find CDF

$$F_Y(y) = Pr(g(X) \le y) = \int_{\{x \mid g(x) \le y\}} f_X(x)\,dx$$

2) Differentiate to get pdf

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$X \sim U[0,1]$ What is pdf of $Y = e^X$?

Note $X > 0$ in $[0,1]$ $\Rightarrow$ $Y > 0$ in $[1,e]$

$$\Pr(Y \le y) = \Pr(e^X \le y)$$
$$= \Pr(X \le \ln(y))$$
$$= \int_0^{\ln y} f(x)\,dx = \ln(y)$$

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{y}$ $\quad 1 \le y \le e$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_1^e y \cdot \frac{1}{y}\,dy = e - 1$$

$$E(Y) = \int_{-\infty}^{\infty} e^x f_X(x)\,dx = \int_0^1 e^x\,dx = e^{-1}$$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

X cont w/ CDF F_X pmf f_X

find density fn of $Y = 2X$

$$F_Y(y) = Pr(Y \leq y)$$
$$= Pr(2X \leq y)$$
$$= Pr\left(X \leq \frac{y}{2}\right)$$
$$= F_X\left(\frac{y}{2}\right)$$

$$\frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y}{2}\right) = \frac{1}{2} f_X\left(\frac{y}{2}\right)$$

To see that this makes sense

$$Pr\left(a-\frac{\varepsilon}{2} \leq X \leq a+\frac{\varepsilon}{2}\right) \approx \varepsilon f(a)$$

$$\varepsilon f_Y(a) \approx Pr\left(a-\frac{\varepsilon}{2} \leq Y \leq a+\frac{\varepsilon}{2}\right)$$
$$= Pr\left(a-\frac{\varepsilon}{2} \leq 2X \leq a+\frac{\varepsilon}{2}\right)$$
$$= Pr\left(\frac{a}{2}-\frac{\varepsilon}{4} \leq X \leq \frac{a}{2}+\frac{\varepsilon}{4}\right)$$
$$\approx \frac{\varepsilon}{2} f_X\left(\frac{a}{2}\right) = \varepsilon \cdot \frac{1}{2} f_X\left(\frac{a}{2}\right)$$

$X_1,\dots,X_n$ iid. $U[0,1]$ What is $E(\max(X_1,\dots,X_n))$?

Let $X = \max(X_1,\dots,X_n)$

$$F_X(x) = \Pr(\max(X_1,\dots,X_n) \le x) = \Pr(X_1 \le x, X_2 \le x, \dots, X_n \le x)$$

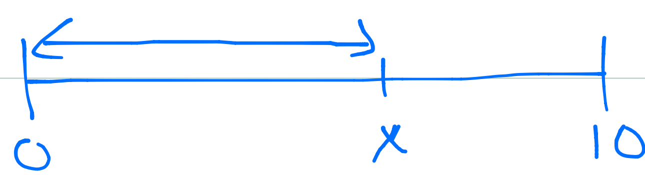
$$= x^n$$

$$f_X(x) = \frac{d}{dx} F_X(x) = n x^{n-1}$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x \cdot n x^{n-1}\,dx = n \int_0^1 x^n\,dx = n \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{n}{n+1}$$

$X \sim \text{Unif}[0..10]$

$Y \sim \text{Unif}[0..X]$



$E(Y) = ?$

$$E(Y) = \int_{-\infty}^{\infty} E(Y \mid X=u) f_X(u)\,du$$

$Y | X=u \sim \text{Unif}[0..u]$

$$E(Y) = \int_0^{10} \frac{u}{2} \frac{1}{10} du = \frac{u^2}{40}\Big|_0^{10} = \frac{100}{40} = 2.5$$

$$\text{or } \frac{1}{2} \int_0^{10} u \frac{1}{10} du = \frac{5}{2} = 2.5$$

$f_{XY}(x,y)$? $\qquad f_Y(y)$?

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X=x}(y)$$

$$= \frac{1}{10} \cdot \frac{1}{x} \qquad 0 \le y \le x \le 10$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

$$= \int_y^{10} \frac{1}{10x} dx = \frac{1}{10} \ln x \Big|_y^{10} = \frac{1}{10} \ln\left(\frac{10}{y}\right) \qquad 0 < y \le 10$$

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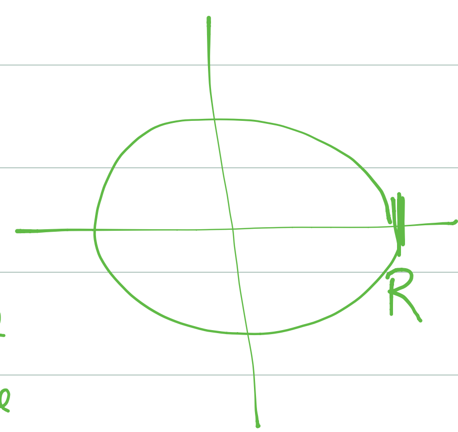
circle of radius R

centered at origin

(X,Y) coordinates of random pt in circle

("dart" equally likely to fall anywhere



$\Rightarrow f(x,y) = \begin{cases} c & x^2+y^2 \le R \\ 0 & \text{o.w.} \end{cases}$

① What is c?

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$c \iint_{x^2+y^2 \le R} dx\,dy = 1$$

use polar coordinates or observe $\iint$ = area of circle

$$\Rightarrow c = \frac{1}{\pi R^2}$$

② What is marginal density of X?

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{\pi R^2} \int_{x^2+y^2 \le R} dy$$

$$= \frac{1}{\pi R^2} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \frac{2}{\pi R^2}\sqrt{R^2-x^2} \qquad x^2 \le R^2$$

$$0 \quad \text{o.w.}$$

③ Pr(distance from origin to pt selected ≤ a) $\qquad D = \sqrt{X^2+Y^2}$

$$F_D(a) = \Pr\left(\sqrt{X^2+Y^2} \le a^2\right)$$

$$= \Pr\left(X^2+Y^2 \le a^2\right)$$

$$= \int_{x^2+y^2 \le a^2} f(x,y)\,dy\,dx$$

$$= \frac{1}{\pi R^2} \underbrace{\int_{x^2+y^2 \le a^2} dy\,dx}_{\text{area of circle of radius } a} \doteq \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}$$

④ $E(D)$

$$f_D(a) = \frac{d}{da} F_D(a) = \frac{2a}{R^2} \quad \text{for } 0 \le a \le R, \quad 0 \text{ o.w.}$$

$$E(D) = \int_0^R a f_D(a) = \frac{2}{R^2} \int_0^R a^2\,da = \frac{2}{3} R$$

$$f(x,y) = x e^{-x(y+1)} \quad x>0,\ y>0$$

What is $f(X \mid Y=y) = \frac{f(x,y)}{f_Y(y)}$? $\quad f_Y(y) = \int_0^\infty x e^{-x(y+1)}\,dx$

Find den ity of. $Z = XY$

$$F_Z(z) = Pr(XY \le z)$$

$$= \int_0^\infty \Pr(XY \le z \mid X = x)\, f_X(x)\, dx$$

$$= \int_0^\infty \Pr\left(Y \le \frac{z}{x} \,\middle|\, X = x\right) f_X(x)\, dx$$

$$= \int_0^\infty F_Y\left(\frac{z}{x}\right) f_X(x)\, dx$$