CSE312 Quiz Section: January 10, 2013

1. How many ways are there to select 5 cards from a standard deck of 52 cards, where the 5 cards contain cards from at most two suits if:

   (a) order does not matter

   $$4 \times \binom{52}{5} + \binom{4}{2} \times \sum_{i=1}^{4} \binom{5}{i} \binom{5}{5-i}$$

   (b) order matters

   $$(4 \times \binom{52}{5} + \binom{4}{2} \times \sum_{i=1}^{4} \binom{5}{i} \binom{5}{5-i}) \times 5!$$

2. Consider all natural numbers between 2 and 18 (both included).

   (a) How many of them are prime?

   $$P = \{2, 3, 5, 7, 11, 13, 17\}$$

   $$|P| = 7$$

   (b) How many of them are prime or of the form $3k-1, k \in N$? Let A be set of number of that form.

   $$A = \{2, 5, 8, 11, 14, 17\}$$

   We want:

   $$|P \cup A| = |P| + |A| - |A \cap P|$$

   $$A \cap P = \{2, 5, 11, 17\}$$

   Therefore:

   $$|P \cup A| = |P| + |A| - |A \cap P| = 7 + 6 - 4 = 9$$

   (c) How many are prime, even or of the form $2^l + 1$, where l is a natural number or zero?

   Consider all three sets and again apply inclusion-exclusion.

   Hint: Inclusion-Exclusion

3. Consider a set of 25 people that form a social network. (The structure of the social network is determined by which pairs of people in the group are "friends".) How many possibilities are there for the structure of this social network?

   A friendship is equivalent to a way of picking two individuals out of the 25. Therefore there are $\binom{25}{2}$ possible friendships. In a social network, each friendship can **independently** exist or not exist. This gives a total of $2^{\binom{25}{2}}$ possible social networks. Alternatively you could consider all binary strings of size $\binom{25}{2}$ and any order on the friendships. Each string encodes a different social network. The i-th bit specifies whether the i-th friendship exists or not. There are 2^l different bitstrings of size l. Here $l = \binom{25}{2}$.

4. At a dinner party, all of the n people present are to be seated at a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down in their correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place.

   Let $r_i, 1 \leq i \leq n$ be the offset of i from his correct seat. These offsets range from 1 to $n-1$, since $0, n$ means that the person is seated correctly, which by assumption cannot be true. Therefore, each offset has $n-1$ possible different values. There are n such offsets (one for each person), that can take $n-1$ possible different values. By the pidgeonhole principle $\exists i, j, i \neq j, \ : \ r_i = r_j$. If I rotate the table by r_i, i and j are seated correctly.

The rest were given as practice problems.

5. Show that:

   (a) $i^2 = 2\binom{i}{2} + \binom{i}{1}$

   (b) $i^3 = 6\binom{i}{3} + 6\binom{i}{2} + \binom{i}{1}$

6. Fix a non-negative integer r. Prove that $\sum_{i=0}^{n} \binom{i}{r} = \binom{n+1}{r+1}$.

7. Use (5) and (6) to show that:

$$\sum_{i=0}^{n} i^2 = 2\binom{n+1}{3} + \binom{n+1}{2}$$

Expand to prove the familiar identity:

$$\sum_{i=0}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$