CSE 312: Quiz Section (/w solutions) Feb 7, 2012

1. A teacher will distribute ten pieces of candy, one at a time, among her ve students. Since she has favorites in the class, she will distribute the candy randomly, where for each piece of candy 1-10, student i has a probability p_i of getting that piece of candy. Let C_i denote the amount of candy that student i ends up with.

   (a) For this to be a valid probabilistic experiment, what do we know about p_1, p_2, p_3, p_4 and p_5?

   $$\sum_{i=1}^{5} p_i = 1$$

   (b) What type of random variable is C_i? Be as specic as possible.

   $$C_i \sim Bin(10, p_i)$$

   (c) Determine $Pr[C_i = k]$ and $E[C_i]$ and $Var[C_i]$. Since it is a binomial distribution:

   $$Pr[C_i = k] = \binom{10}{k} {p_i}^k (1 - p_i)^{10-k}$$

   $$E[C_i] = 10p_i$$

   $$Var[C_i] = 10p_i(1 - p_i)$$

   (d) If all p_i are equal, find $Pr[Max(C_1, C_2, C_3, C_4, C_5) \leq 2]$.

   We only need to account for $max = 2$ since by the pidgeonhole principle at least one student will get at least 2 pieces of candy. But, if $max = 2$ then every $C_i = 2$. Therefore we need to find $Pr[C_1 = C_2 = C_3 = C_4 = C_5 = 2]$. This follows a multinomial distribution. Intuitively I can specify $\binom{10}{2}$ ways to give two pieces to the first guy. Then $\binom{8}{2}$ ways for the second and so forth. Moreover, the candy will be distributed this way with probability ${p_1}^2{p_2}^2{p_3}^2{p_4}^2{p_5}^2 = 2^{-10}$. Overall:

   $$Pr[Max(C_1, C_2, C_3, C_4, C_5) \leq 2] = Pr[C_1 = C_2 = C_3 = C_4 = C_5 = 2] =$$

   $$\binom{10}{2}\binom{8}{2}..\binom{2}{2}2^{-10} = \frac{10!}{2!2!2!2!2!}2^{-10}$$

2. You have an opportunity to bet on a number between 1 and 6. Three dice are then rolled. If your number fails to appear, you lose 1\$. If it appears once, you win 1\$; if twice, 2\$; if three times, 3\$. How much are you willing to pay to play this game?

We have to find $E[Gain]$:

$$E[Gain] = -(\frac{5}{6})^3 + \binom{3}{1}(\frac{1}{6})^1(\frac{5}{6})^2 + 2\binom{3}{2}(\frac{1}{6})^2(\frac{5}{6})^1 + 3\binom{3}{3}(\frac{1}{6})^3(\frac{5}{6})^0 =$$

I can add a "free" term $0\binom{3}{0}(\frac{1}{6})^0(\frac{5}{6})^3$ (since it is equal to zero) and I get:

$$-(\frac{5}{6})^3 + \sum_{i=0}^{3}\binom{3}{i}(\frac{1}{6})^i(\frac{5}{6})^{3-i}$$

The second term is the expectation of a binomial distribution:

$$E[Gain] = -(\frac{5}{6})^3 + \frac{3}{6} = \frac{1}{2} - (\frac{5}{6})^3$$

So they would have to pay me to play the game.

3. When using a fair 6-sided die, what is the expected number of 1s you see before you get the first 6?

   Let N, S be the number of ones you see before your first six and the trial at which you saw your first six. Then by the law of total expectation:

$$E[N] = E[E[N|S]] = \sum_{i=0}^{\infty} E[N|S = i]Pr[S = i]$$

   But $N|S = i$ follows a binomial distribution with parameters $i - 1$ and $\frac{1}{5}$, therefore $E[N|S = i] = \frac{1}{5}(i - 1)$:

$$E[N] = \sum_{i=0}^{\infty} \frac{1}{5}(i - 1)Pr[S = i] = \frac{1}{5}(\sum_{i=0}^{\infty} iPr[S = i] - \sum_{i=0}^{\infty} Pr[S = i])$$

   The first sum is the expectation of a geometric distribution with parameter $\frac{1}{6}$ and is equal to $\frac{1}{\frac{1}{6}} = 6$ and the second sum is one, since it is the sum over all individual probabilities. Therefore:

$$E[N] = \frac{1}{5}(6 - 1) = 1$$

4. Suppose that for Halloween, everyone gets a brown bag with two pieces of candy. Each piece is chosen randomly to be either chocolate or vanilla (so all possibilities CC, CV, VC, VV are equally likely). You reach into the bag and pull out one of the two candies at random. If you pull out a chocolate candy, whats the probability that the other candy is vanilla?

The probability is $\frac{1}{2}$. It is the same as if you flipped two coins independently and somebody is asking what is the probability that you get heads in the second flip if you got tails on the first. Since their are independent, it doesn't matter what the first flip was ; the probability is still $\frac{1}{2}$.

5. How many ways are there to fill k slots by choosing from n elements while allowing repeat? (If I am not allowing repeat, its just n choose k)

   It is:

$$\binom{n+k-1}{k}$$

   This is similar to the "stars and bars" kind of problem that you had in one of your past homework.