Midterm Review

CSE 312

Counting

- **Product Rule:** If there are n outcomes for some event A, sequentially followed by m outcomes for event B, then there are $n \bullet m$ outcomes overall. General: $n1 \times n2 \times ... \times nk$
- **Permutation**: an arrangement of objects in a definite order $N!/(N-n)!$
- **Combination**: a selection of objects with no regard to order $N!/[n!(N-n)!]$

Binomial Theorem

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

Inclusion-Exclusion

- for two sets or events A and B, whether or not they are disjoint, $|A \cup B| = |A| + |B| - |A \cap B|$

- General: $|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap C| - |A \cap B| + |A \cap B \cap C|$

Pigeonhole Principle

- If there are n pigeons in k holes and n > k, then some hole contains more than one pigeon. More precisely, some hole contains at least $\lceil n/k \rceil$ pigeons.

- Problem: network problem on HW

Sample spaces / Events / Sets

- **Sample space:** S is the set of all possible outcomes of an experiment (notation: Ω)
- **Events:** $E \subseteq S$ is an arbitrary subset of the sample space
- **Set:**

  subset: $A \subset B$

  Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

  Intersection: $A \cap B = \{x \in A \text{ and } x \in B\}$

  Complement: $A' = \{x \mid x \notin A\} = A\text{^}c$

  Mutually Exclusive / Disjoint: $A \cap B = \emptyset$

  Any number of sets A1,A2,A3,...are mutually exclusive if and only if $Ai \cap Aj = \emptyset$ for $i \neq j$

DeMorgan's Laws

$$\overline{E \cup F} = \bar{E} \cap \bar{F}$$

$$\overline{E \cap F} = \bar{E} \cup \bar{F}$$

Axioms of Probability

- Axiom 1 (Non-negativity): $0 \leq \Pr(E)$
- Axiom 2 (Normalization): $\Pr(S) = 1$
- Axiom 3 (Additivity): If E and F are mutually exclusive ($EF = \emptyset$), then $\Pr(E \cup F) = \Pr(E) + \Pr(F)$

If events E1, E2, …En are mutually exclusive

$$\Pr\left(\bigcup_{i=1}^{n} E_i\right) = \Pr(E_1) + \cdots + \Pr(E_n)$$

Conditional Probability

- **Conditional probability** of E given F: probability that E occurs given that F has occurred. P(E|F)

$$P(E \mid F) = \frac{|EF|}{|F|} = \frac{|EF|/|S|}{|F|/|S|} = \boxed{\frac{P(EF)}{P(F)}}$$

Chain Rule

- $$P(E \mid F) = \frac{P(EF)}{P(F)}$$ where, P(F) > 0

- General definition of Chain Rule:

$$P(E_1 E_2 \cdots E_n) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1, E_2) \cdots P(E_n \mid E_1, E_2, \ldots, E_{n-1})$$

Law of Total Probability

- E and F are events in the sample space S:

  E = EF U EF'

  P(E) = P(EF) + P(EFc)

  = P(E|F) P(F) + P(E|Fc) P(Fc)

  = P(E|F) P(F) + P(E|Fc) (1-P(F))

  P(E) = ∑i P(E|Fi) P(Fi)

Bayes Theorem

Most common form:

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E)}$$

Expanded form (using law of total probability):

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid F^c)P(F^c)}$$

Proof:

$$P(F \mid E) = \frac{P(EF)}{P(E)} = \frac{P(E \mid F)P(F)}{P(E)}$$

Independence

- Two events E and F are independent if

P(EF) = P(E)P(F). If P(F) > 0, P(E|F) = P(E)
Otherwise, they are dependent.

- Three events E, F, G are independent if

P(EF) = P(E)P(F) P(EG) = P(E)P(G) P(FG) = P(G)P(G) and P(EFG) = P(E)P(F)P(G)

- Events E1, E2, …, En are independent if for every subset S of {1,2,…, n}, we have

$$P\left(\bigcap_{i \in S} E_i\right) = \prod_{i \in S} P(E_i)$$

Independence

- Theorem: E, F independent $\Rightarrow$ E, F' independent

- Theorem: if $P(E)>0$, $P(F)>0$, then

E, F independent $\Leftrightarrow P(E|F)=P(E) \Leftrightarrow P(F|E) = P(F)$

Network Failure

- **Parallel**: n routers in parallel, ith has probability pi of failing, independently

P(there is functional path) = 1 – P(all routers fail) = 1 – p1p2 … pn

- **Series:** n routers, ith has probability pi of failing, independently

P(there is functional path) = P(no routers fail) = (1 – p1)(1 – p2) … (1 – pn)

Conditional Independence

- Two events E and F are called conditionally independent given G, if
- $P(EF|G) = P(E|G)\,P(F|G)$

- Or, $P(E|FG) = P(E|G)$, $(P(F)>0,\ P(G)>0)$

PMF / CDF

- PMF: probability mass function

$$p(a) = \begin{cases} P(X = a) & \text{for } a \in T \\ 0 & \text{otherwise} \end{cases}$$

- CDF: cumulative distribution function:

$$F(a) = P[X \leq a]$$

$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \leq a < 2 \\ \frac{3}{4} & 2 \leq a < 3 \\ \frac{7}{8} & 3 \leq a < 4 \\ 1 & 4 \leq a \end{cases}$$

Expectation

- For a discrete r.v. X with p.m.f. p(•), the expectation of X (expected value or mean), is $E[X] = \Sigma_x \, xp(x)$

Properties of Expectation

- **Linearity**:
- For any constants a, b: $E[aX + b] = aE[X] + b$

- Let X and Y be two random variables derived from outcomes of a single experiment. Then $E[X+Y] = E[X] + E[Y]$

Variance

- The variance of a random variable X with mean $E[X] = \mu$ is $Var[X] = E[(X-\mu)^2]$, often denoted σ^2.

Properties of Variance

- 1. $\mathrm{Var}(X) = E[X^2] - (E[X])^2$
- 2. Var[aX+b] = a^2 * Var[X]
- 3. Var[X+Y] ≠ Var[X] + Var[Y]

r.v.s Independence

- Defn: Random variable X and event E are independent if the event E is independent of the event $\{X=x\}$ (for any fixed x), i.e. $\forall x\ P(\{X = x\}\ \&\ E) = P(\{X=x\}) \cdot P(E)$

- Defn: Two random variables X and Y are independent if the events $\{X=x\}$ and $\{Y=y\}$ are independent (for any fixed x, y), i.e.

$$\forall x, y\ P(\{X = x\}\ \&\ \{Y=y\}) = P(\{X=x\}) \cdot P(\{Y=y\})$$

Joint Distributions

- **Joint probability mass function:**

fXY(x, y) = P({X = x} & {Y = y})

- **Joint cumulative distribution function:**

FXY(x, y) = P({X ≤ x} & {Y ≤ y})

Marginal Distributions

- Marginal PMF of one r.v.: sum over the other

- $f_Y(y) = \Sigma_x f_{XY}(x,y)$
- $f_X(x) = \Sigma_y f_{XY}(x,y)$

Discrete Random Variables

Bernoulli Distribution

Definition: value 1 with probability p, 0 otherwise (prob. $q = 1-p$)

Example: coin toss ($p = ½$ for fair coin)

Parameters: p

Properties:

$E[X] = p$

$Var[X] = p(1-p) = pq$

Binomial Distribution

Definition: sum of n independent Bernoulli trials, each with parameter p

Example: number of heads in 10 independent coin tosses

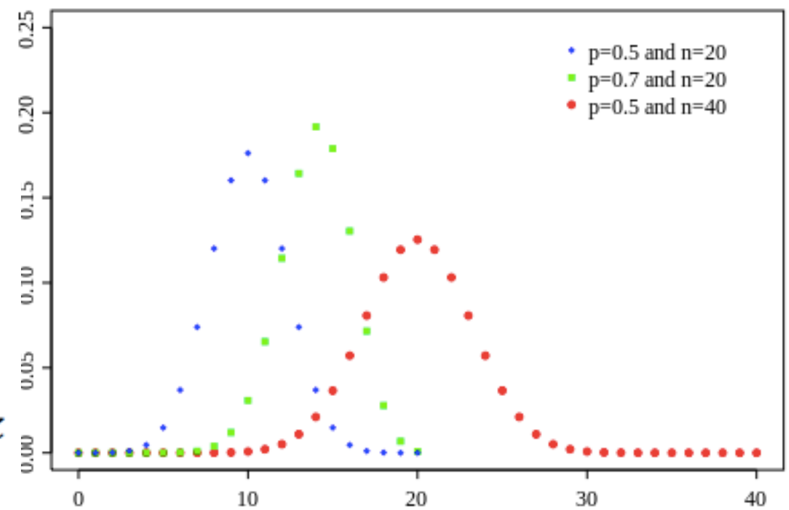
Parameters: n, p

Properties:

$E[X] = np$

$Var(X) = np(1-p)$

pmf: $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$



Poisson Distribution

Definition: number of events that occur in a unit of time, if those events occur independently at an average rate λ per unit time

Example: # of cars at traffic light in 1 minute, # of deaths in 1 year by horse kick in Prussian cavalry

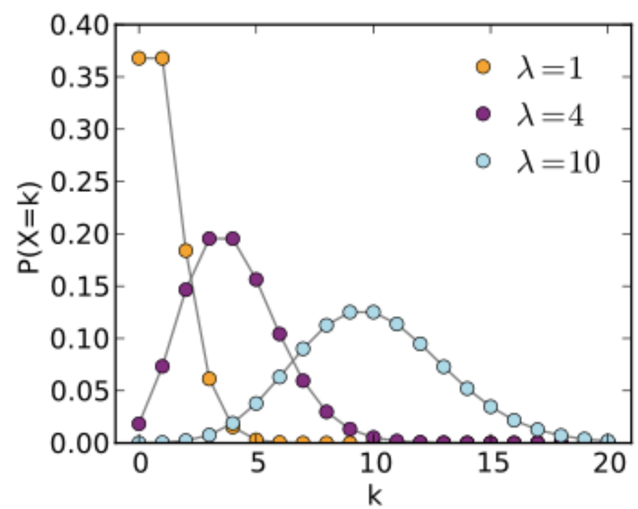
Parameters: λ

Properties:

$E[X] = \lambda$

$Var[X] = \lambda$

pmf: $Pr(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$



Geometric Distribution

Definition: number of independent Bernoulli trials with parameter p until and including first success (so X can take values 1, 2, 3, ...)

Example: # of coins flipped until first head

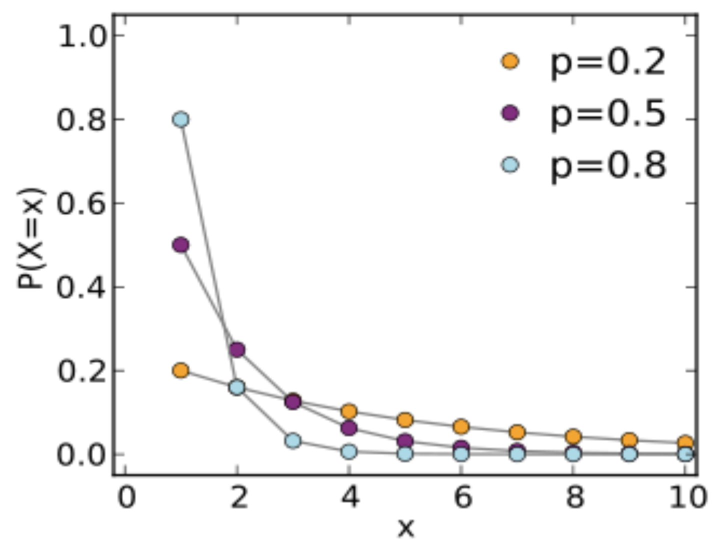
Parameters: p

Properties:

$$E[X] = \frac{1}{p}$$

$$\text{Var}[X] = \frac{1-p}{p^2}$$

pmf: $\Pr(X = k) = (1-p)^{k-1}p$



Hypergeometric Distribution

Definition: number of successes in *n* draws (without replacement) from *N* items that contain *K* successes in total

Example: An urn has 10 red balls and 10 blue balls. What is the probability of drawing 2 red balls in 4 draws?

Parameters: *n, N, K*

Properties:

$$\text{E[X]} = n\frac{K}{N}$$

$$\text{Var[X]} = n\frac{K}{N}\frac{N-K}{N}\frac{N-n}{N-1}$$

pmf: $$Pr(X=k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Think about the pmf; we've been doing it for weeks now: ways-to-choose-successes times ways-to-choose-failures over ways-to-choose-n

Also, consider that the binomial dist. is the with-replacement analog of this