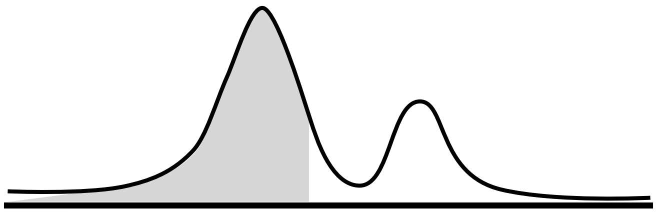
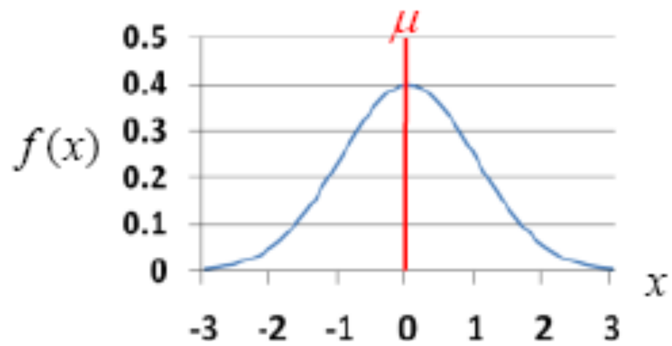
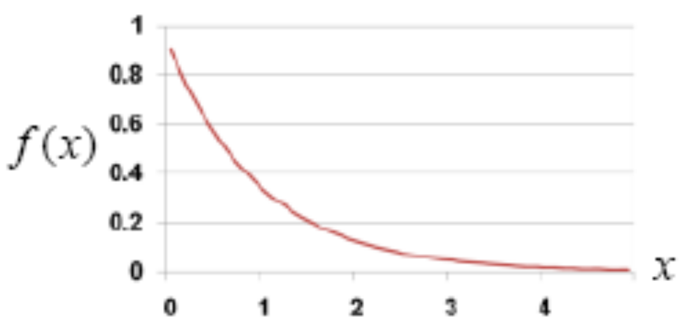
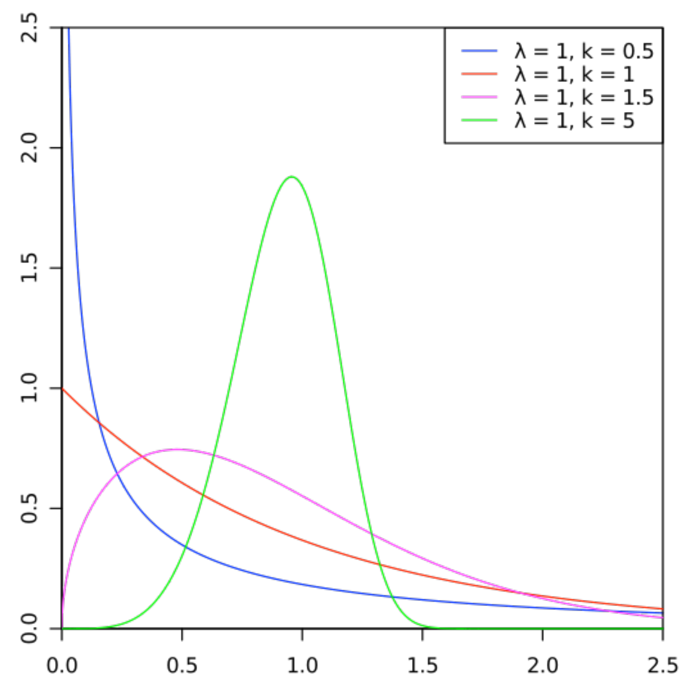




continuous random variables





Discrete random variable: takes values in a finite or countable set, e.g.

$X \in \{1,2, ..., 6\}$ with equal probability

X is positive integer i with probability 2^{-i}

Continuous random variable: takes values in an uncountable set, e.g.

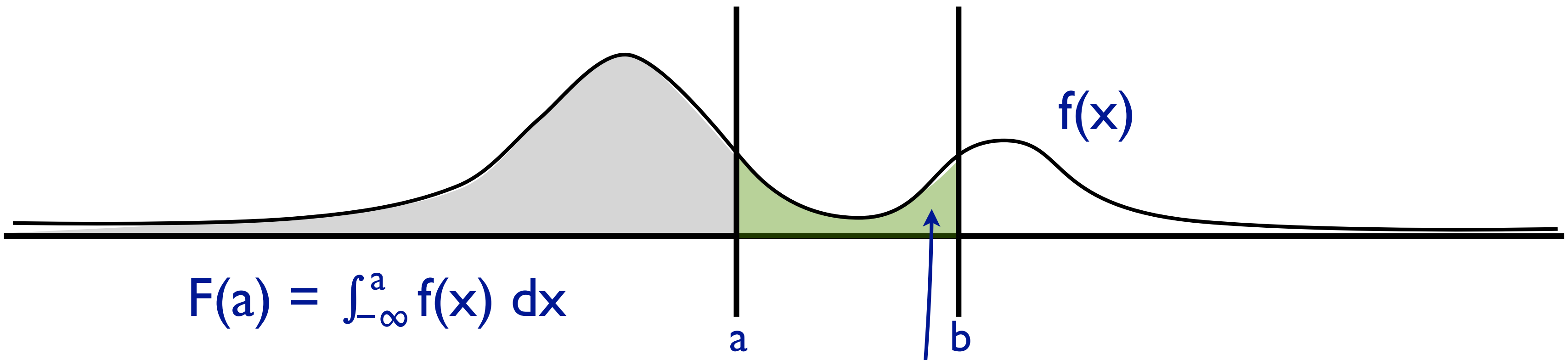
X is the weight of a random person (a real number)

X is a randomly selected point inside a unit square

X is the waiting time until the next packet arrives at the server

pdf and cdf

f(x) : the *probability density function* (or simply "density")



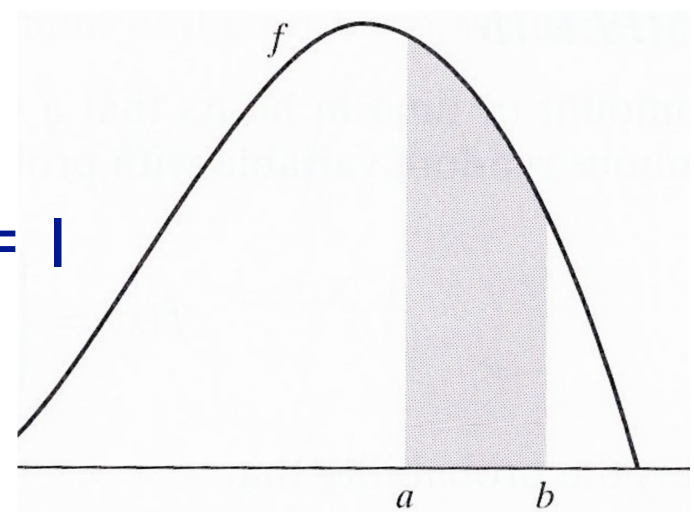
P(X ≤ a) = F(x): the *cumulative distribution function* (or simply "distribution")

P(a < X ≤ b) = F(b) - F(a)

Need f(x)≥0, & $\int_{-\infty}^{+\infty} f(x)\,dx$ (= F(+∞)) = 1

A key relationship:

$f(x) = \frac{d}{dx} F(x)$, since $F(a) = \int_{-\infty}^{a} f(x)\,dx$,



Densities are *not* probabilities

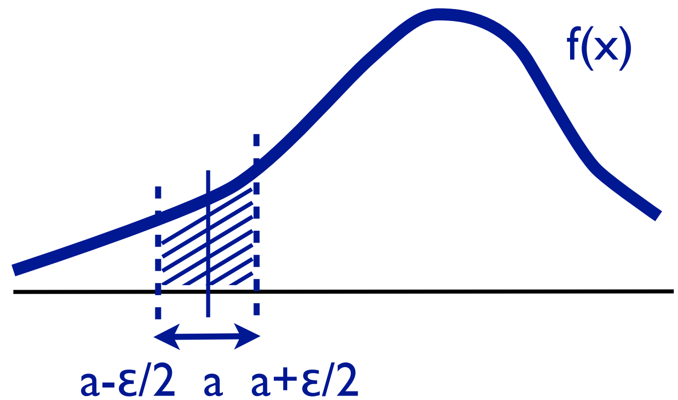
$P(x = a) = P(a \le X \le a) = F(a)\text{-}F(a) = 0$

I.e., the probability that a continuous random variable falls *at* a specified point is *zero*

$P(a - \varepsilon/2 \le X \le a + \varepsilon/2) =$

$F(a + \varepsilon/2) - F(a - \varepsilon/2)$

$\approx \varepsilon \cdot f(a)$



I.e., The probability that it falls *near* that point is proportional to the density; in a large random sample, expect more samples where density is higher (hence the name "density").

Much of what we did with discrete r.v.s carries over almost unchanged, with Σ_x... replaced by $\int$... dx

E.g.

For discrete r.v. X, $E[X] = \Sigma_x \, xp(x)$

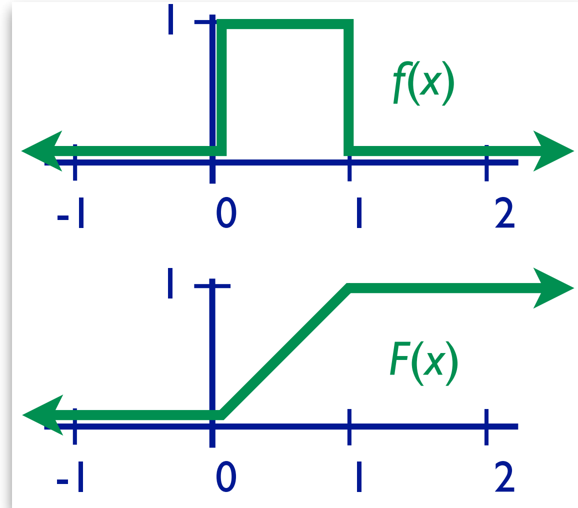
For continuous r.v. X, $E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

Why?

(a) We define it that way

(b) The probability that X falls "near" x, say within $x \pm dx/2$, is $\approx f(x)dx$, so the "average" X should be $\approx \Sigma\, xf(x)dx$ (summed over grid points spaced dx apart on the real line) and the limit of that as $dx \to 0$ is $\int xf(x)dx$

Let $f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$



$$F(a) = \int_{-\infty}^{a} f(x)dx$$

$$= \begin{cases} 0 & \text{if } a \le 0 \\ a & \text{if } 0 < a \le 1 \text{ (since } a = \int_0^a 1dx) \\ 1 & \text{if } 1 < a \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)dx = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$\text{Var}[X] = E[X^2] - (E[X])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (\sigma \approx 0.29)$

Linearity

$E[aX+b] = aE[X]+b$

$E[X+Y] = E[X]+E[Y]$

still true, just as for discrete

Functions of a random variable

$E[g(X)] = \int g(x)f(x)dx$

just as for discrete, but w/integral

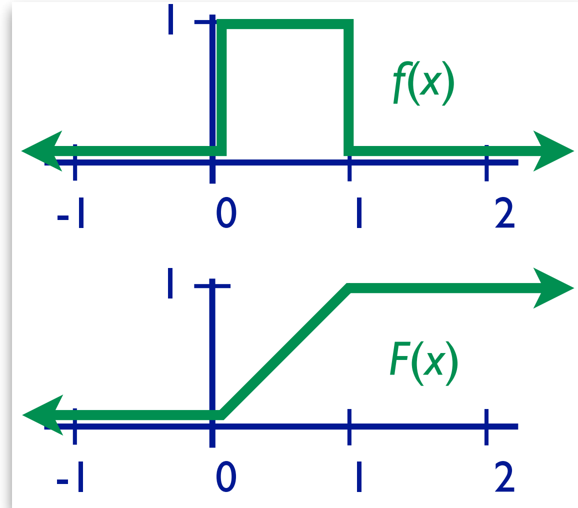
Definition is same as in the discrete case

$Var[X] = E[(X-\mu)^2]$ where $\mu = E[X]$

Identity still holds:

$Var[X] = E[X^2] - (E[X])^2$ proof "same"

Let $f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$



$$F(a) = \int_{-\infty}^{a} f(x)dx$$

$$= \begin{cases} 0 & \text{if } a \le 0 \\ a & \text{if } 0 < a \le 1 \text{ (since } a = \int_0^a 1dx) \\ 1 & \text{if } 1 < a \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)dx = \int_0^1 x\, dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$$\text{Var}[X] = E[X^2] - (E[X])^2 = \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12} \quad (\sigma \approx 0.29)$$

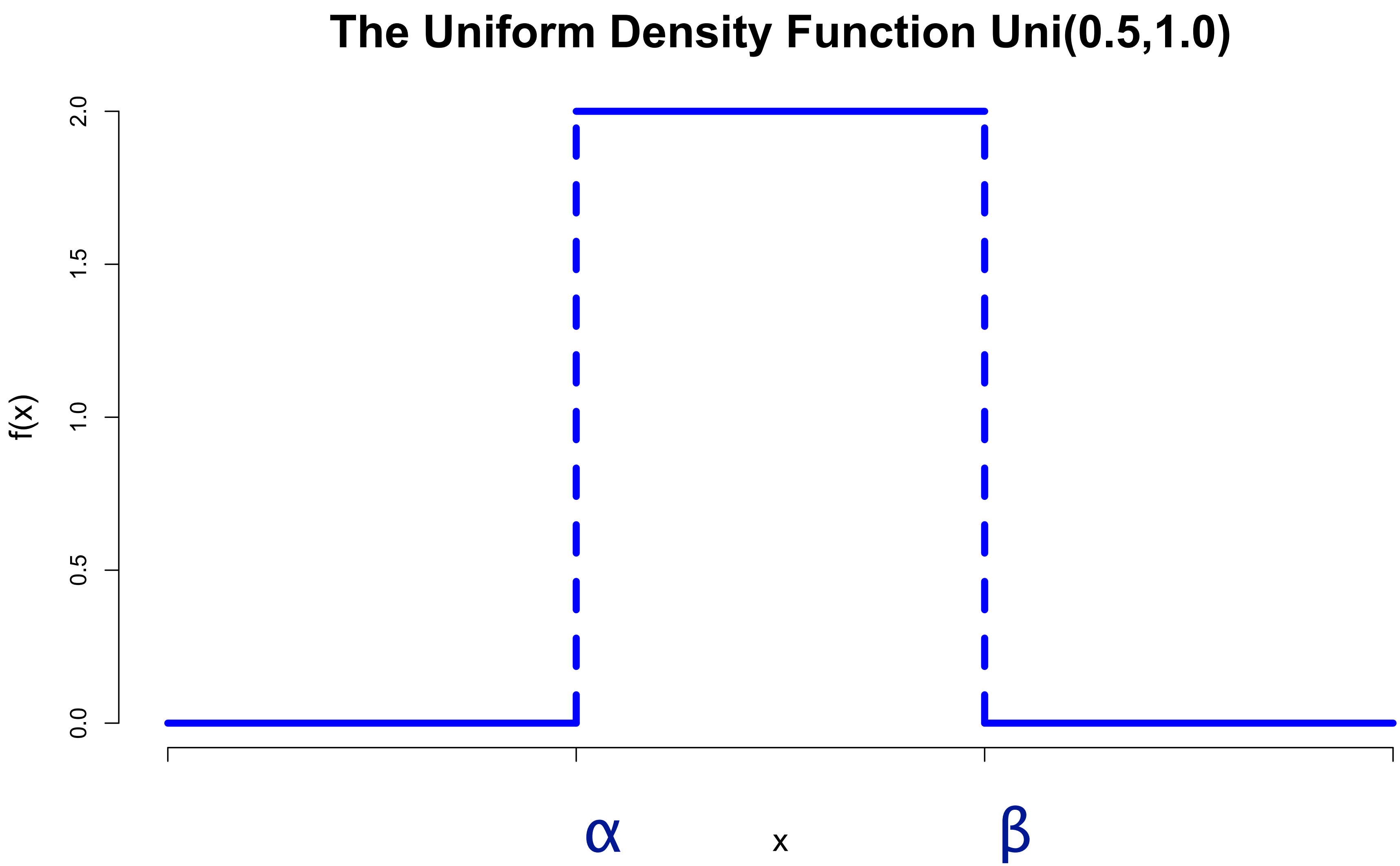
Continuous random variable X has density f(x), and

$$\Pr(a \leq X \leq b) = \int_a^b f(x)\,dx$$

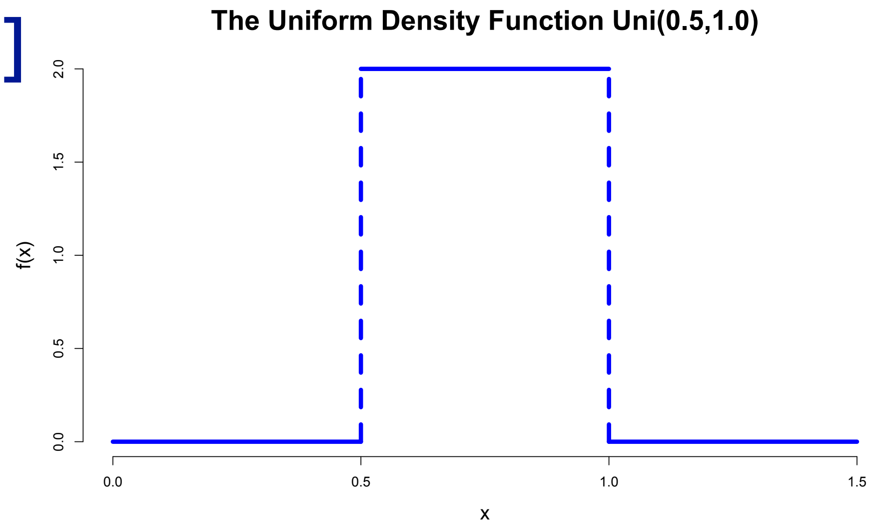
$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$$

X ~ Uni(α,β) is uniform in [α,β] $f(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in [\alpha,\beta] \\ 0 & \text{otherwise} \end{cases}$



X ~ Uni(α,β) is uniform in [α,β]



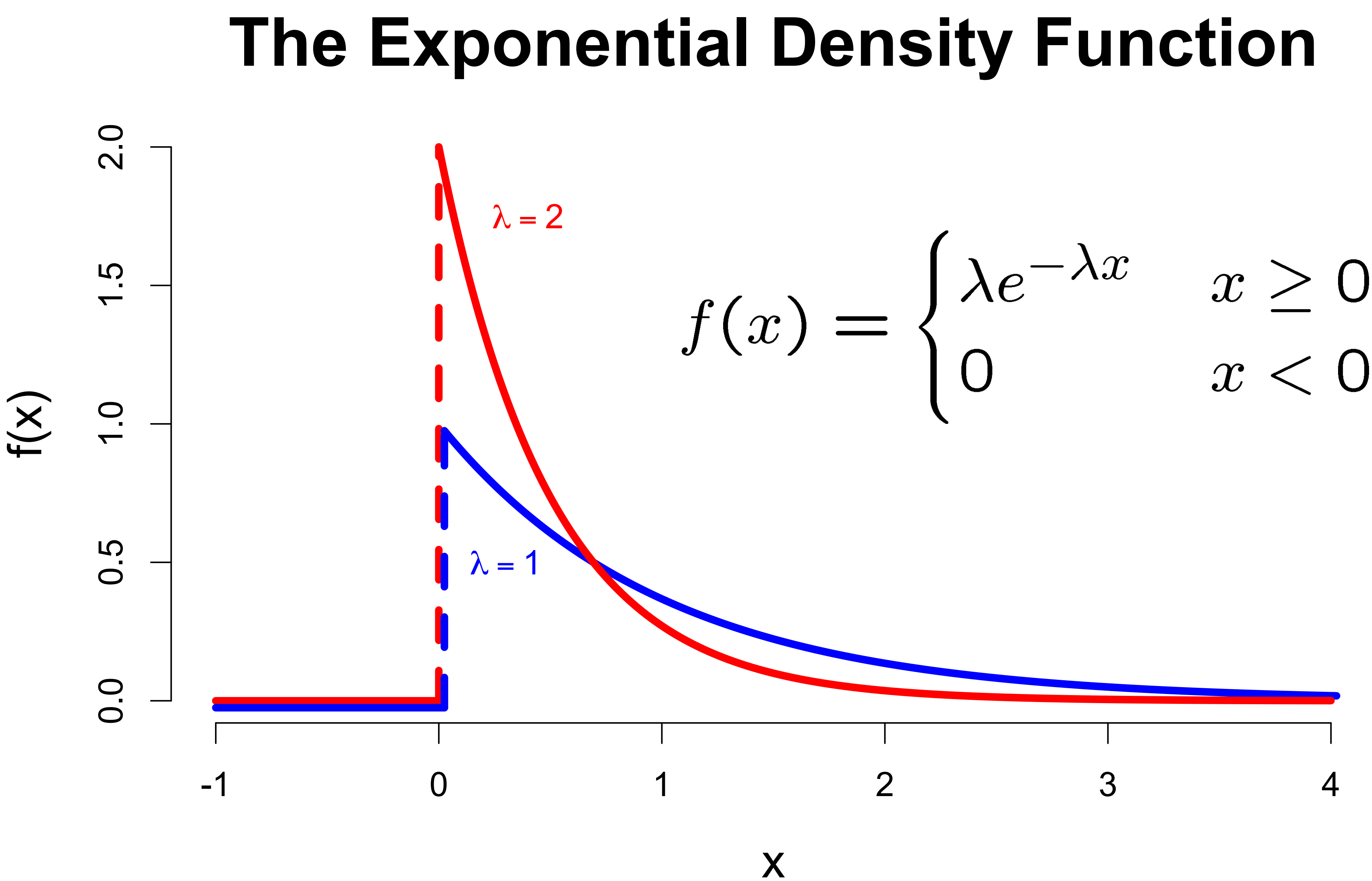
$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in [\alpha, \beta] \\ 0 & \text{otherwise} \end{cases}$$

$$\Pr(a \le X \le b) = \int_a^b f(x)\,dx = \frac{b-a}{\beta-\alpha}$$

if $\alpha \le a \le b \le \beta$:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \frac{\alpha+\beta}{2}$$

$X \sim Exp(\lambda)$



X ~ Exp(λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda} \qquad \mathrm{Var}[X] = \frac{1}{\lambda^2}$$

$$\Pr(X \geq t) = e^{-\lambda t} = 1\text{-}F(t)$$

Memorylessness:

$$\Pr(X > s + t \mid X > s) = \Pr(X > t)$$

Radioactive decay: How long until the next alpha particle?

Customers: how long until the next customer/packet arrives at the checkout stand/server?

Buses: How long until the next #71 bus arrives on the Ave?

Yes, they have a schedule, but given the vagaries of traffic, riders with-bikes-and-baby-carriages, etc., can they stick to it?

Relation to the Poisson:

Poisson: *how many* events in a *fixed time*;
Exponential: *how long* until the *next event*

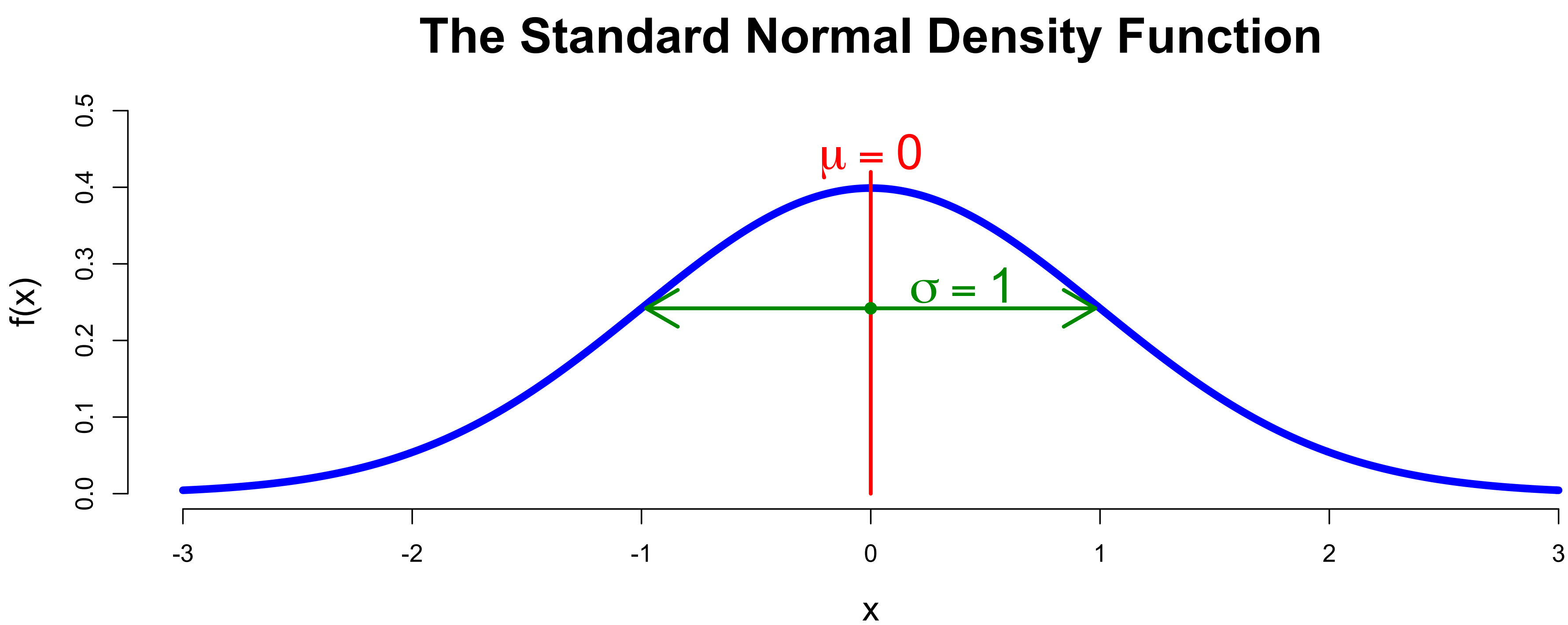
Relation to geometric: Geometric is discrete analog:

How long to a Head, 1 flip per sec, prob p vs
How long to a Head, 2 flips per sec, prob p/2, ...
Limit is exponential with parameter p

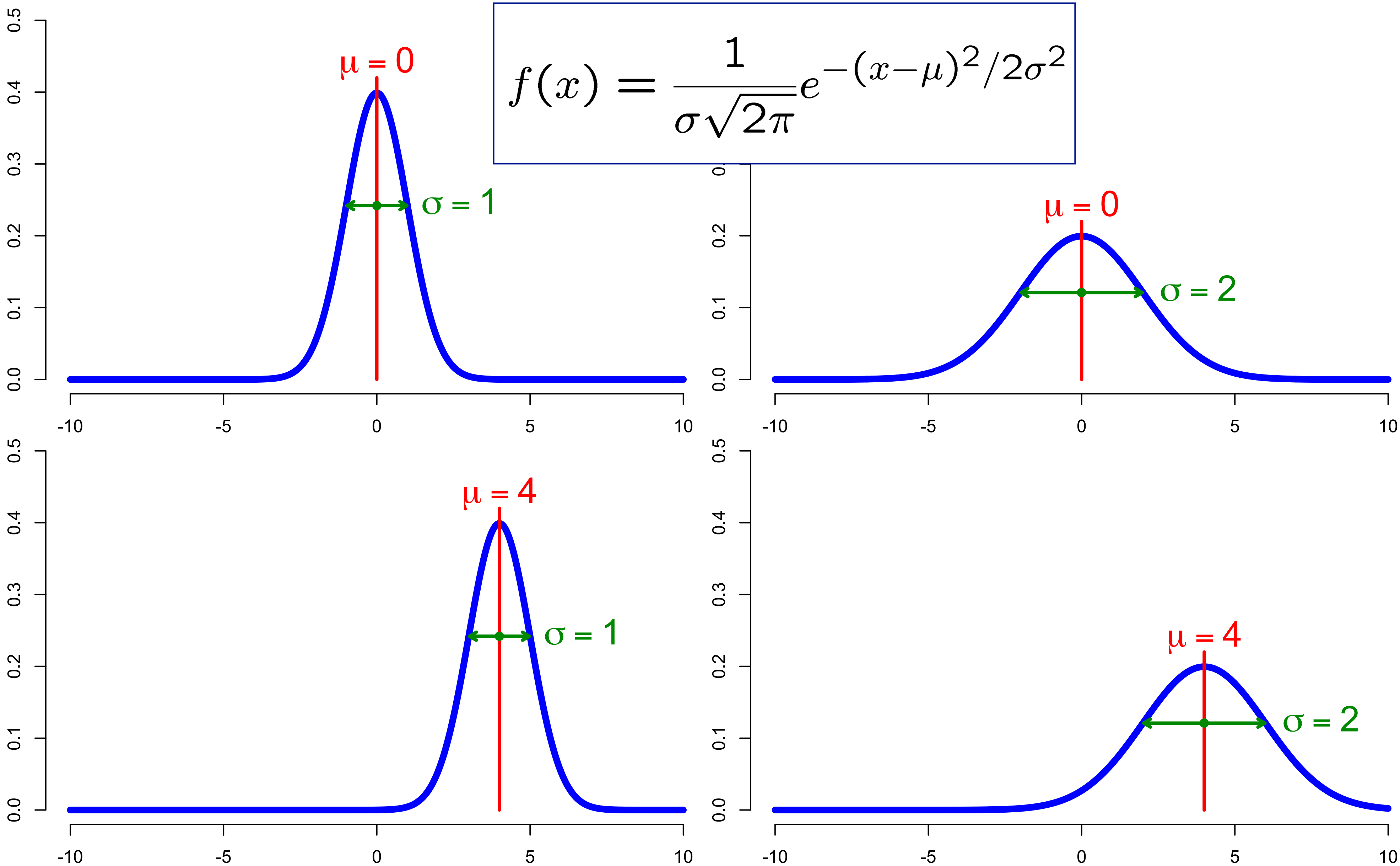
X is a normal (aka Gaussian) random variable $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu \qquad \mathrm{Var}[X] = \sigma^2$$



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$



density at μ is ≈ .399/σ

X is a normal random variable $X \sim N(\mu,\sigma^2)$

$Y = aX + b$

$E[Y] = E[aX+b] = a\mu + b$

$Var[Y] = Var[aX+b] = a^2\sigma^2$

$Y \sim N(a\mu + b, a^2\sigma^2)$

Important special case: $Z = (X-\mu)/\sigma \sim N(0,1)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$Z \sim N(0,1)$ "*standard* (or *unit*) normal"

Use $\Phi(z)$ to denote CDF, i.e.

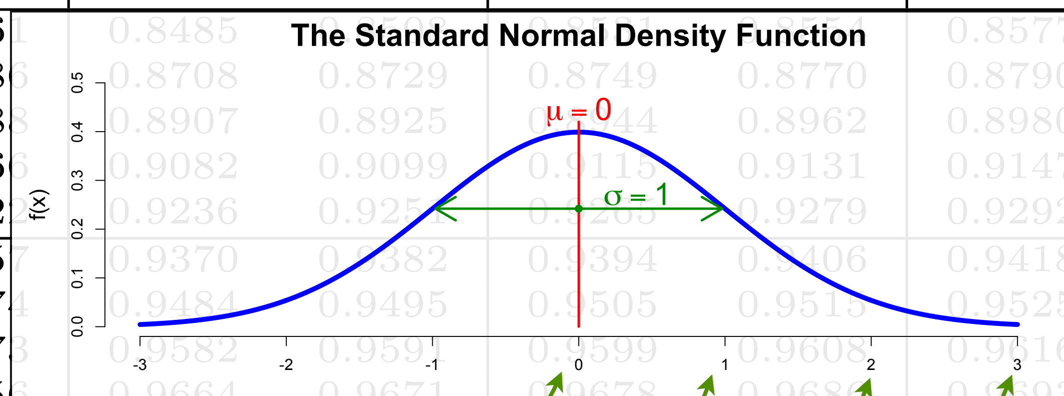
$$\Phi(z) = \Pr(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

no closed form ☹

Table of the Standard Normal Cumulative Distribution Function $\Phi(Z)$

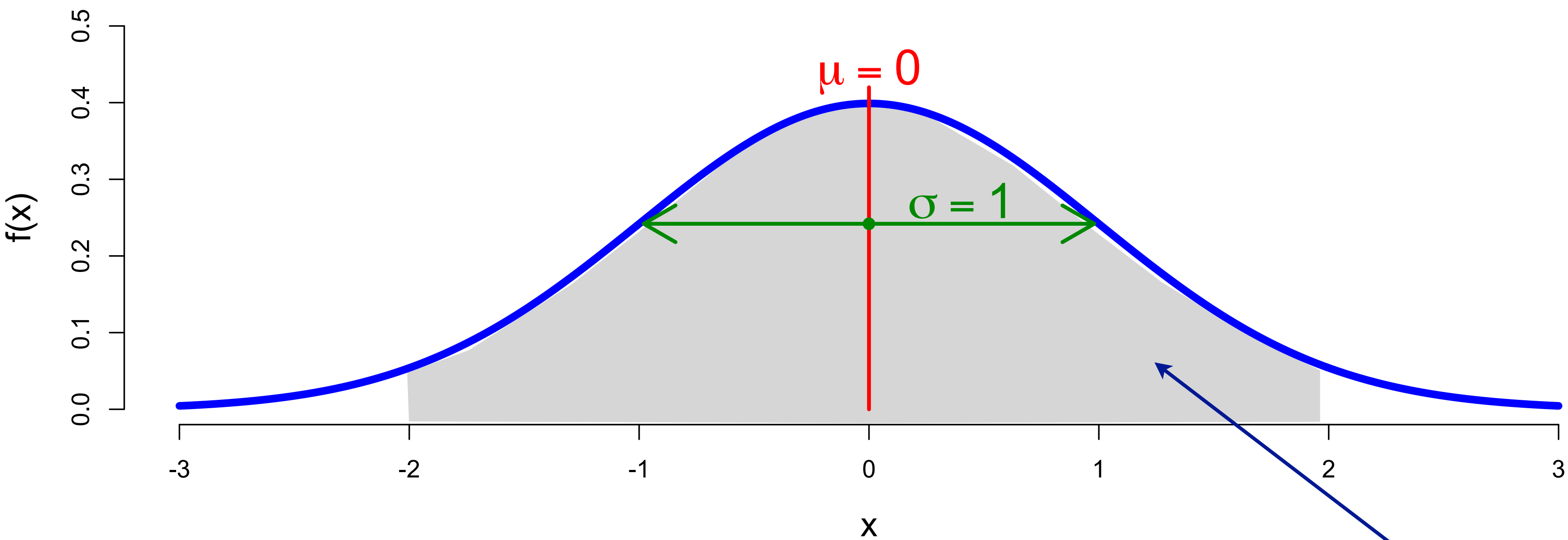
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	[illegible]	0.7517
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	[illegible]	0.7823
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365
1.0	0.8413	0.8438	0.8461	0.8485	[illegible]	[illegible]	[illegible]	0.8577	0.8599
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	[illegible]	[illegible]	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	[illegible]	[illegible]	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997

$\Phi(.46)$



E.g., see B&T p155, p531

The Standard Normal Density Function



If Z ~N(μ,σ) what is P(μ-σ < Z < μ+σ)?

$P(\mu - \sigma < Z < \mu + \sigma) = \Phi(1) - \Phi(-1) \approx 68\%$

$P(\mu - 2\sigma < Z < \mu + 2\sigma) = \Phi(2) - \Phi(-2) \approx 95\%$

$P(\mu - 3\sigma < Z < \mu + 3\sigma) = \Phi(3) - \Phi(-3) \approx 99\%$

X ~ Bin(n,p)

E[X] = np   Var[X] = np(1-p)

Poisson approx: good for n large, p small (np constant)

Normal approx: For large n, (p stays fixed):

$X \approx Y \sim N(E[X], Var[X]) = N(np, np(1-p))$

Normal approximation good when $np(1-p) \geq 10$

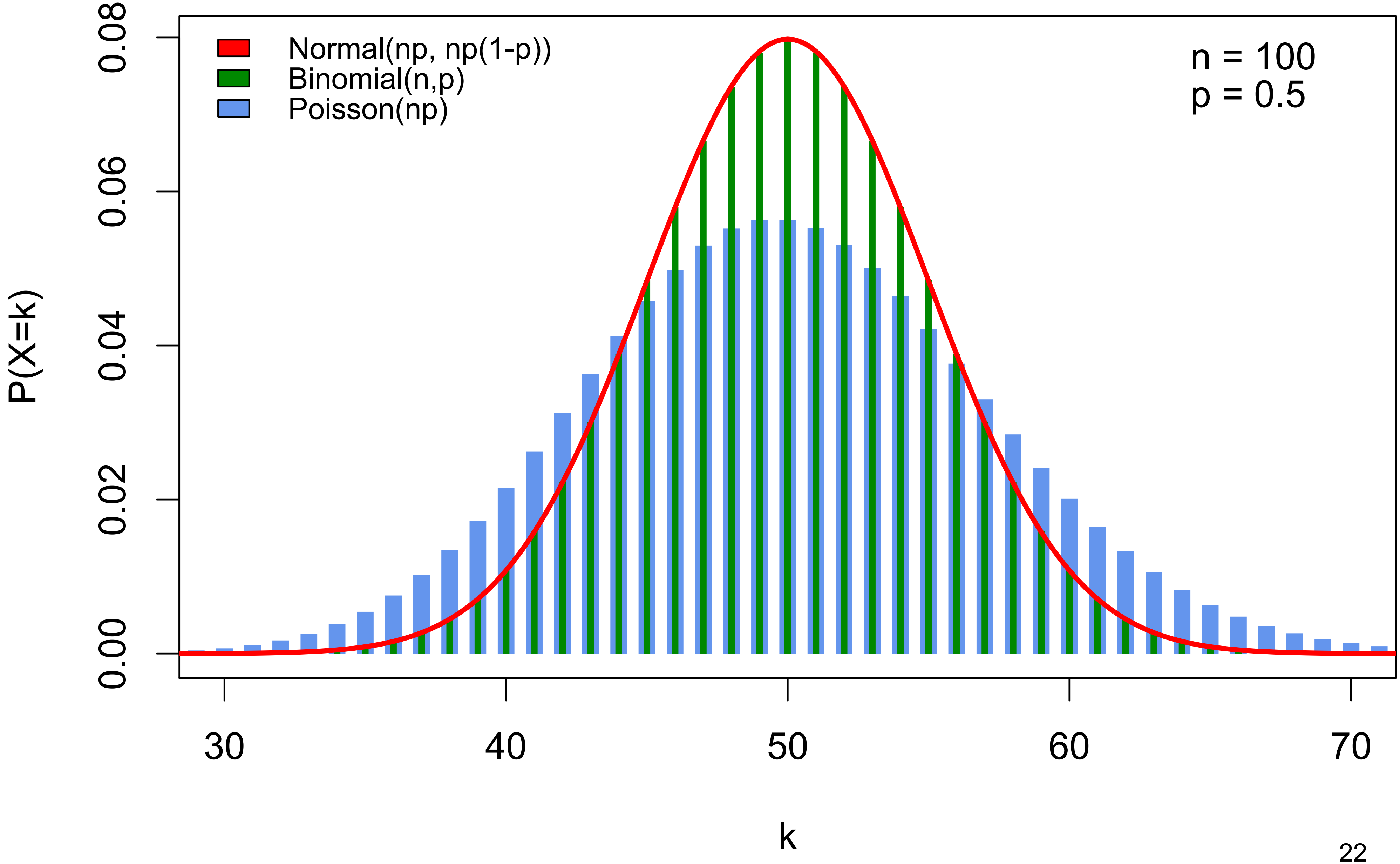
DeMoivre-Laplace Theorem:

Let S_n = number of successes in n trials (with prob. p).

Then, as $n \to \infty$:

$$Pr\left(a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right) \longrightarrow \Phi(b) - \Phi(a)$$

normal approximation to binomial



Fair coin flipped 40 times. Probability of 20 heads?

Exact answer:

$$P(X = 20) = \binom{40}{20}\left(\frac{1}{2}\right)^{40} \approx \boxed{0.1254}$$

Normal approximation:

$$\begin{aligned} P(X = 20) &= P(19.5 \le X < 20.5) \\ &= P\left(\frac{19.5-20}{\sqrt{10}} \le \frac{X-20}{\sqrt{10}} < \frac{20.5-20}{\sqrt{10}}\right) \\ &\approx P\left(-0.16 \le \frac{X-20}{\sqrt{10}} < 0.16\right) \\ &\approx \Phi(0.16) - \Phi(-0.16) \approx \boxed{0.1272} \end{aligned}$$

the central limit theorem (CLT)

Consider i.i.d. (independent, identically distributed) random vars $X_1, X_2, X_3, \ldots$

X_i has $\mu = E[X_i]$ and $\sigma^2 = Var[X_i]$

As $n \to \infty$,

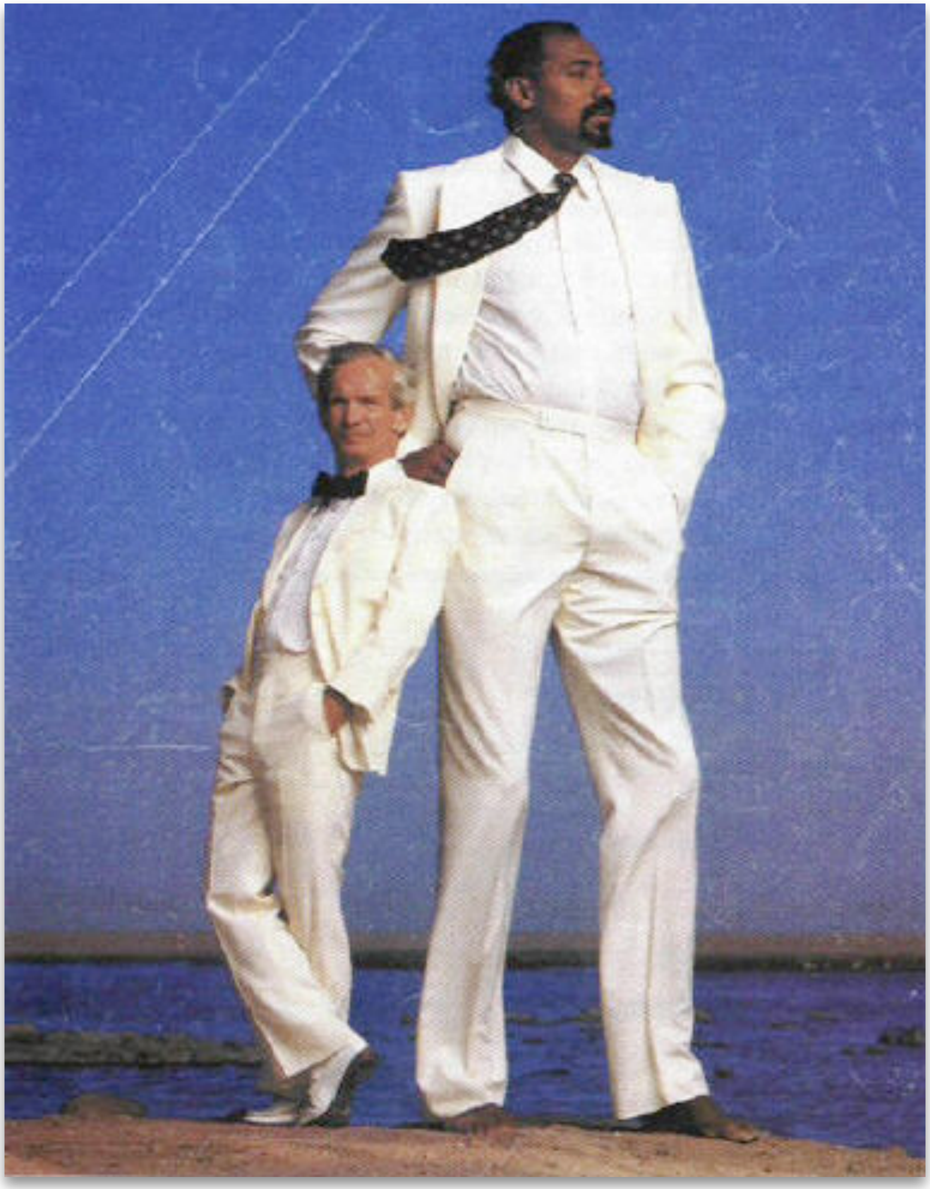
$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \longrightarrow N(0,1)$$

Restated: As $n \to \infty$,

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

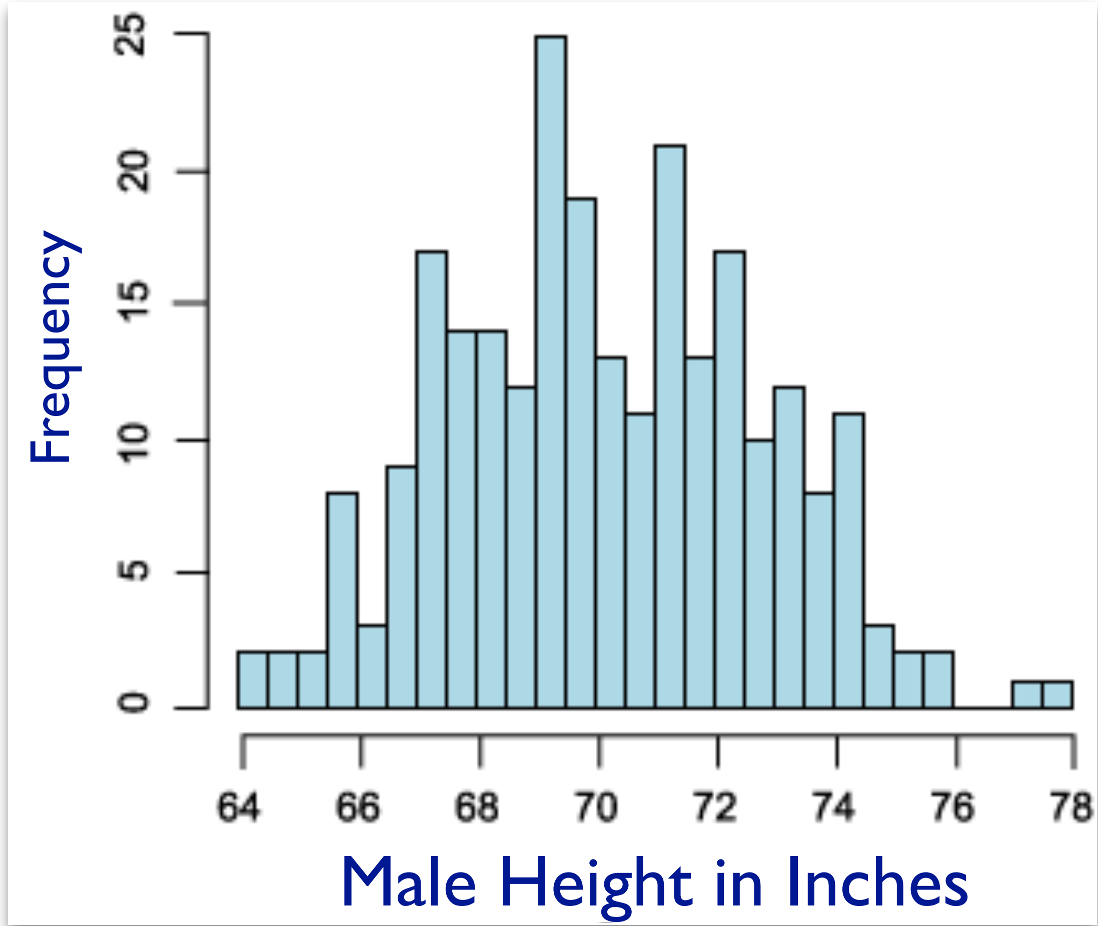
How tall are you? Why?

Credit: Annie Leibovitz, © 1987 ?



Willie Shoemaker & Wilt Chamberlain

Human height is approximately normal.



Why might that be true?

R.A. Fisher (1918) noted it would follow from CLT if height were the sum of many independent random effects, e.g. many genetic factors (plus some environmental ones like diet). I.e., suggested part of *mechanism* by looking at *shape* of the curve. (WAY before anyone really knew what genes were...)

Meta-analysis of Dense Genecentric Association Studies Reveals Common and Uncommon Variants Associated with Height, Lanktree, et al.

The American Journal of Human Genetics **88**, 6–18, January 7, 2011

12 The American Journal of Human Genetics 88, 6–18, January 7, 2011

Table 1. Sixty-Four Loci Showing Significant Evidence for Association with Adult Height, Identified with the Use of the IBC Array

(and hundreds more probably exist)

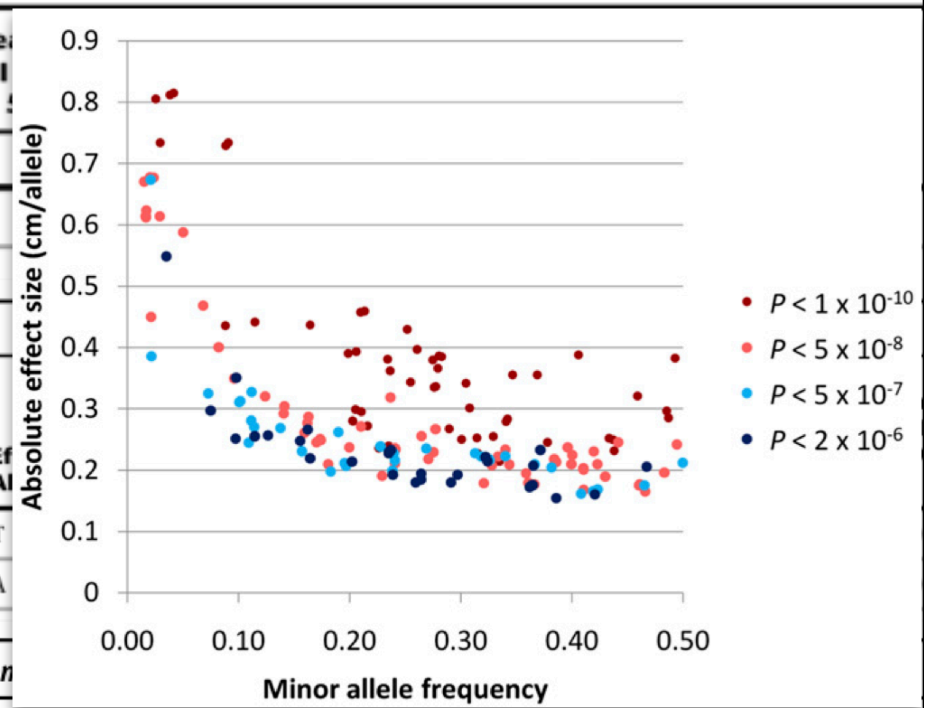
Locus Rank	Chr.	Candidate Gene[a]	SNP[a]	Effect Allele	MAF	European Phase I (up to ... Effect
1	7q22	*CDK6*	rs4272	A	0.21	−0.46
2	6p21	*HMGA1*	rs1150781	C	0.09	0.73
3	12q15	*HMGA2*				
4	20q11	*MMP24*				
5	17q23	*MAP3K3*				
6	17q24	*GH1-GH2*				
7	1p36	*MFAP2*				
8	15q26	*IGF1R*				
9	7p22	*GNA12*				
10	17q23	*TBX2*				
11	12q22	*SOCS2*				
12	9q22	*PTCH1*				
13	14q11	*NFATC4*				
14	15q26	*ACAN*				
15	2q24	*NPPC*				
16	6p21	*PPARD*				
17	20q11	*MYH7B*				
18	19q13	*IL11*				

Table 1. *Continued*

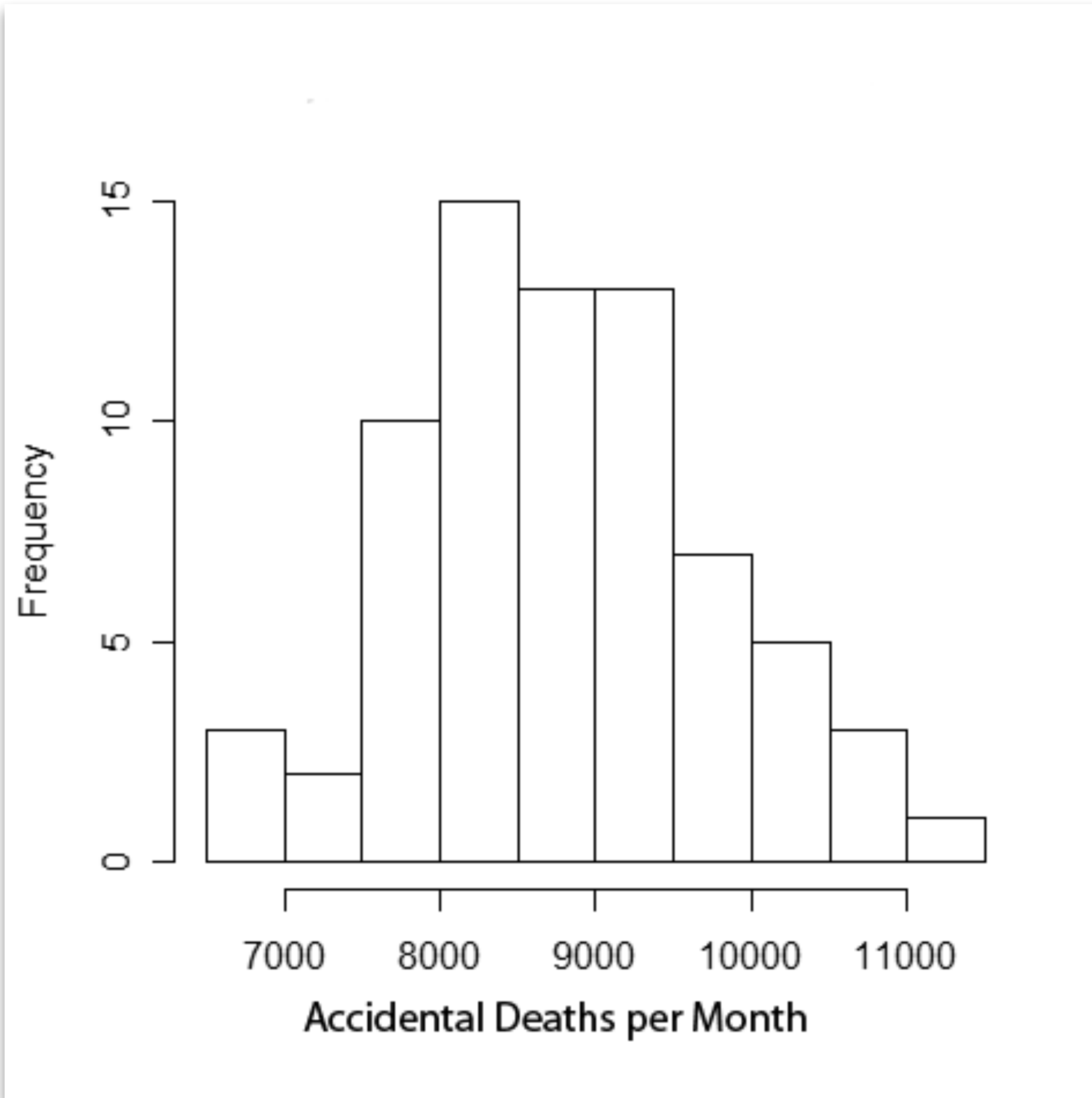
Locus Rank	Chr.	Candidate Gene[a]	SNP[a]	Ef Al
28	2p23	*GCKR*	rs780094	T
29	1q41	*TGFB2*	rs900	A
30	20q11	*CDK5RAP1*		
31	2p12	*EIF2AK3*		
32	19p13	*INSR*		
33	6q25	*ESR1*		
34	2q37	*DIS3L2*		
35	2q35	*PLCD4*		
36	1p36	*RPS6KA1*		
37	15q21	*CYP19A1*		
38	5q31	*SLC22A5*		
39	7p15	*JAZF1*		
40	17p13	*POLR2A*		
41	1p22	*PKN2*		
42	7q22	*CNOT4*		

Table 1. Con

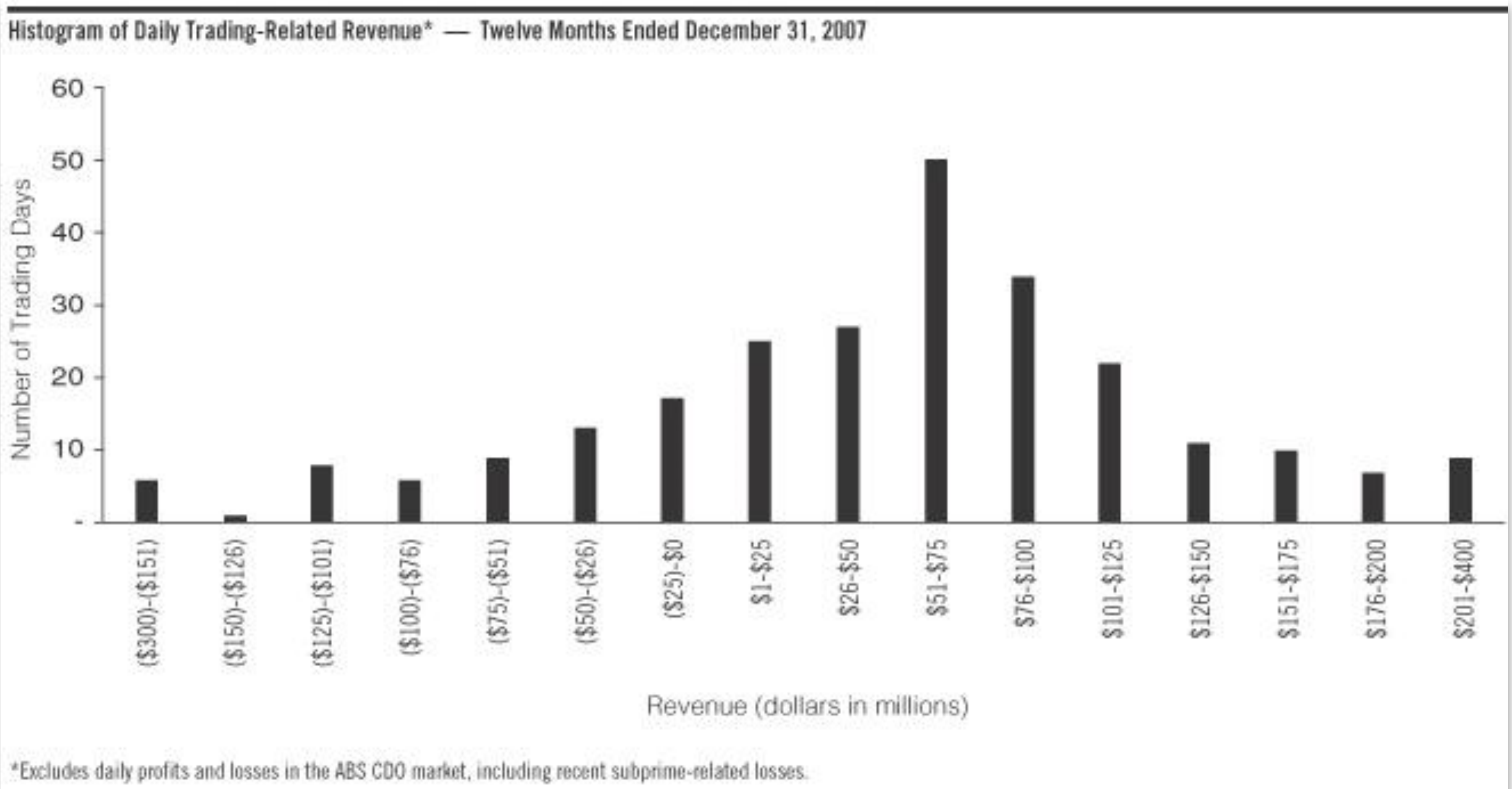
Locus Rank	Chr.	Candidate Gene[a]	SNP[a]	Effect Allele	MAF	European Ancestry Phase I (up to 53,394) Effect	p
54	1p22	*COL24A1*	rs2046159	A	0.16	0.23	3.8×10^{-5}
55	1q23	*DUSP23*	rs1129923	A	0.10	−0.25	2.7×10^{-4}
56	10q22	*MAT1A*	rs7087728	A	0.18	0.22	2.2×10^{-4}
57	2p15	*PPP3R1*	rs1822469	T	0.41	−0.14	7.8×10^{-4}
58	7q36	*ATG9B*	rs1800783	A	0.38	−0.16	2.0×10^{-4}
59	14q11	*BCL2L2*	rs3210043	A	0.16	0.25	9.7×10^{-6}
60	4p14	*RFC1*	rs11096991	T	0.35	0.15	3.6×10^{-4}

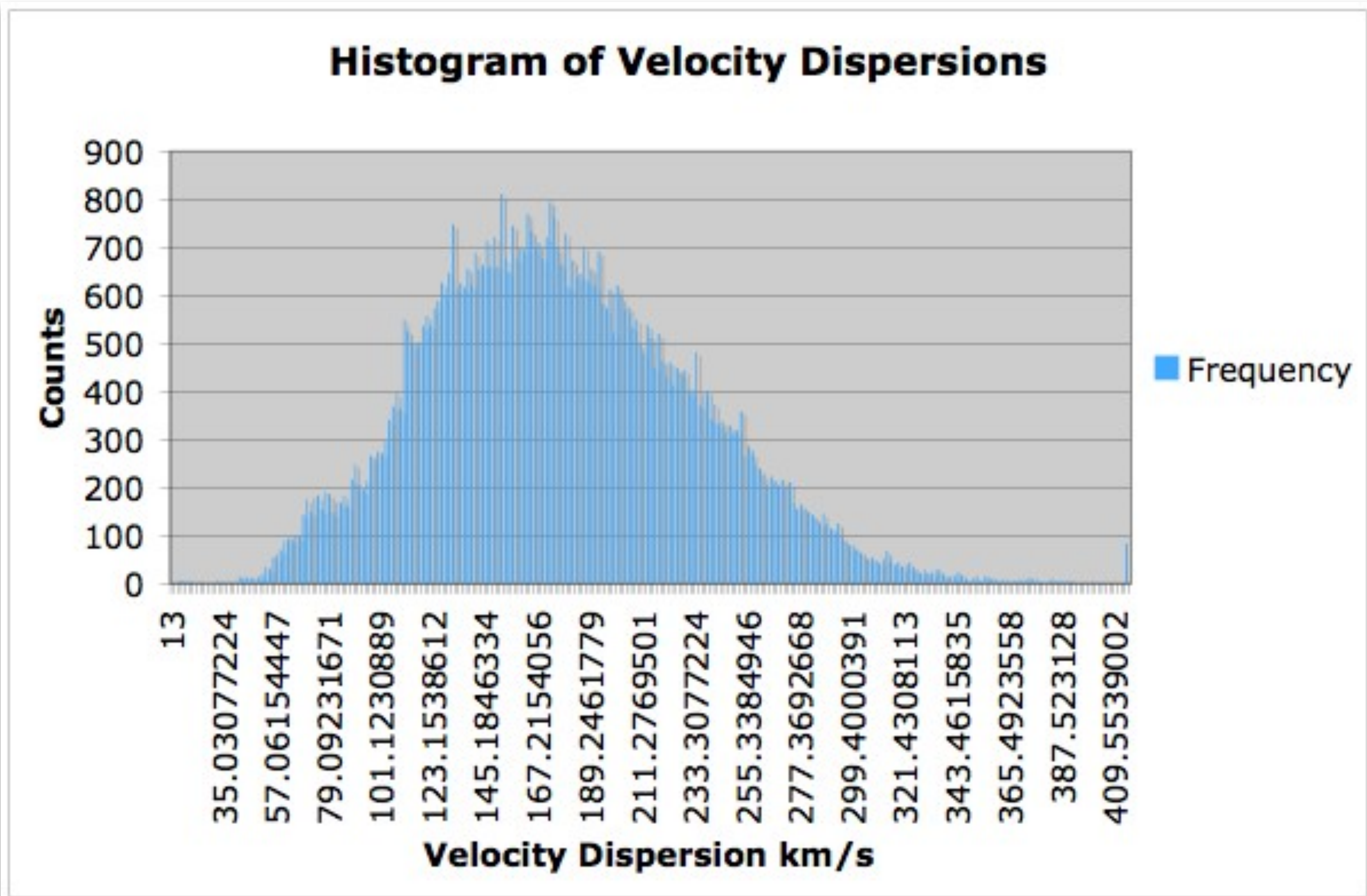


in the real world…



in the real world…




Histogram of Velocity Dispersions
Counts
900
800
700
600
500
400
300
200
100
0
Frequency
13
35.030772224
57.061544447
79.092231671
101.12308889
123.15386612
145.18463334
167.21540056
189.24617779
211.27695501
233.30772224
255.33849946
277.36926668
299.40003391
321.43081113
343.46158835
365.49235558
387.5231280
409.55390002
Velocity Dispersion km/s

pdf and cdf

$f(x) = \frac{d}{dx} F(x)$ $F(a) = \int_{-\infty}^{a} f(x)\,dx$

sums become integrals, e.g.

$E[X] = \Sigma_x\, xp(x)$ $$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

most familiar properties still hold, e.g.

$E[aX+bY+c] = aE[X]+bE[Y]+c$

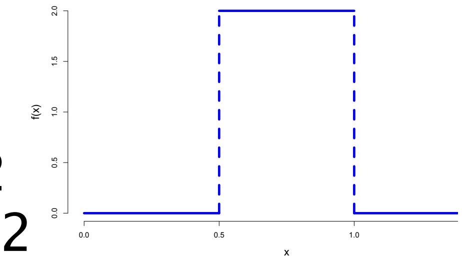
$Var[X] = E[X^2] - (E[X])^2$

Three important examples

X ~ Uni(α,β) uniform in [α,β]

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in [\alpha, \beta] \\ 0 & \text{otherwise} \end{cases}$$

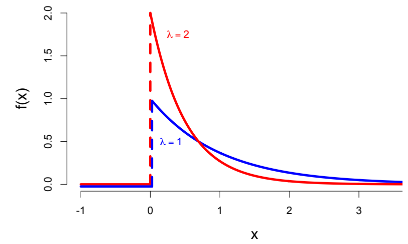
E[X] = (α+β)/2
Var[X] = $(\alpha-\beta)^2/12$



X ~ Exp(λ) exponential

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda}$$
$$\mathrm{Var}[X] = \frac{1}{\lambda^2}$$



X ~ N(μ, σ²) normal (aka Gaussian)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu$$
$$\mathrm{Var}[X] = \sigma^2$$