Exercises

1. Consider the process of rolling three six-sided dice. Represent the outcomes as abc where each a, b, and c can take values 1-6, where (die #1 $= a$), (die #2 $= b$), (die #3$= c$)

   (a) Determine the sample space Ω and calculate $|\Omega|$. For an event abc, what is $\mathbf{Pr}[abc]$?

   (b) Consider the function $f : \Omega \to \mathbb{N}$, where for the event $x = abc \in \Omega$ we define f by

   $$f(x) = \text{the sum of the three dice} = a + b + c.$$

   What is $f(y)$ for $y = 334 \leftrightarrow$ (die #1 $= 3$), (die #2 $= 3$), (die #3$= 4$)? [super easy...]

   (c) Consider the set $A = \{x \mid f(x) = 10\}$. Describe this set in words, and determine $|A|$ (the easiest way to do this is by brute force).

   (d) Thinking abstractly, what if we wanted to associate probabilities with different values of f? How would you determine $\mathbf{Pr}[f(x) = 10]$? To interpret this, think about rolling the three dice, and then computing $f(x)$ based on the outcome. What is the probability of getting the value $f(x) = 10$?

2. To save space when computing the probabilities based on a function, let's define an object called a random variable. Formally, a *random variable* X is a function from the sample space to the real numbers. Instead of the typical $f(x)$ notation, since we only will care about the probabilities, we will write $X = x$ for the *event* that an outcome occurs which causes X to take the value x.

   (a) Explain how random variables relate to the definition of an event as a subset of the sample space.

   (b) Consider the process of flipping a fair coin five times in a row. Let Y be the random variable which maps a sequence of five coin flips to the number heads in the outcome. Calculate $\mathbf{Pr}[Y = k]$ for all $k = 0, 1, \ldots, 5$. What about for $k = 6$?

   (c) We can also make sense of the event $Y \geq k$ in the natural way – it means the event that you flip the five coins and you get at least k heads. Calculate $\mathbf{Pr}[Y \geq 2]$.

3. Define the *expected value* $\mathbf{E}[X]$ (which is a real valued function of a random variable X) as

   $$\mathbf{E}[X] = \sum_{i=1}^{m} x_i \cdot \mathbf{Pr}[X = x_i].$$

   where in this case X only takes values in $\{x_1, \ldots, x_m\}$.

   (a) Determine the meaning of the expression "real-valued function of a random variable." In particular, why is it true that the expected value is a function of functions?

   (b) Calculate $\mathbf{E}[Y]$ for Y described in (2b).

   (c) Returning to (1), let X be the random variable which equals the maximum value of the three dice. Calculate $\mathbf{Pr}[X = k]$ for $k = 1, 2, 3, 4, 5, 6$. Calculate $\mathbf{E}[X]$.

Challenge: Random Rooks.

1. Imagine that eight rooks are randomly placed on a standard 8×8 chessboard, where at most one rook can occupy a single space. Find the probability that all the rooks will be safe from one another, *i.e.* that there is no row or column with more than one rook.

2. As n goes to infinity, what does the probability tend to? What does this mean?

3. Consider a three dimensional $8 \times 8 \times 8$ chessboard. Let's label the $3D$ board by triples (i, j, k) where each of i, j, k range from 1 to 8. To generalize safety, we say that two rooks are safe from each other if there are no two dimensions in which both of these rooks' positions agree. (in the $2D$ case, it was if any one dimension agreed).

   If eight rooks are randomly placed on a $8 \times 8 \times 8$ chessboard, what's the probability they will be safe from each other?

Hint: Last section we talked about a sequential method of calculating probabilities, where you imagine each part of the process happening in sequence, and multiply the probabilities at each step.

Hint: There's a natural recursive structure to the problem. After placing one rook on an 8×8 chessboard...