Combinatorics is built upon a collection of very simple identities. Today, you will be going through these identities and really *understanding* why they make sense. Interestingly, you will prove these statements logically, by showing that both sides of the equation count **the same set of elements**. Be forewarned: this course is only going to get harder, so mastering the fundamentals early on will save you a whole lot of time later. (In fact, you should memorize the *derivation/proof* of *everything* on this page.)

Lecture

1. Let $\binom{n}{k}$ denote the number of ways to choose k distinct elements from a set of size n. Explain why $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
2. Prove using *Counting Two Ways* that $\sum_{i=0}^{n} i = \binom{n+1}{2}$.

Exercises

Use **Counting Two Ways**. That is, show that both the left and right sides of the equals sign counts the same set of elements. (Two sides of the equations... two ways to count... get it?).

1. $\binom{n}{k} = \binom{n}{n-k}$.
2. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
3. $i^2 = 2\binom{i}{2} + \binom{i}{1}$ and $i^3 = 6\binom{i}{3} + 6\binom{i}{2} + \binom{i}{1}$. *Hint: Consider ordered pairs/triples.*
4. $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.
5. Fix a nonnegative integer r (imagine $r = 2$ if you want). Prove $\sum_{i=0}^{n} \binom{i}{r} = \binom{n+1}{r+1}$.
6. Use (3) and (5) to prove

$$\sum_{i=0}^{n} i^2 = 2\binom{n+1}{3} + \binom{n+1}{2}.$$

Now, expand the binomial coefficients to prove the familiar (?) identity

$$\sum_{i=0}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

Do at home before your homework!

When permuting n objects, some of which are indistinguishable, different permutations may lead to indistinguishable object sequences, so the number of indistinguishable object sequences is less than $n!$. For example, there are $3!$ permutations of the letters A, B, and C:

$$ABC,\ ACB,\ BAC,\ BCA,\ CAB,\ CBA,$$

but only three distinguishable sequences that can be formed using A, D, and D:

$$ADD,\ DAD,\ DDA.$$

1. Suppose that k out of the n objects are indistinguishable. Show that the number of distinguishable object sequences is $n!/k!$.
2. Suppose that we have r types of indistinguishable objects, and for each i, there are k_i objects of type i. Show that the number of distinguishable object sequences is

$$\frac{n!}{k_1! \cdot k_2! \cdots k_r!}.$$

3. Explain how this reduces to the binomial coefficient $\binom{n}{k}$ when there are only two types of objects.

Challenge problem (one of the coolest combinatorics facts/proofs!)

Determine the number of *integer* and *nonnegative* solutions to the equation

$$x_1 + x_2 + \ldots + x_k = n.$$

To get started, consider the following picture

$$\bullet\bullet \mid \bullet\bullet\bullet\bullet \mid \mid \mid \bullet \mid \bullet\bullet \mid$$

which corresponds to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 9,$$

with values

$$x_1 = 2,\ \ x_2 = 4,\ \ x_3 = 0,\ \ x_4 = 0,\ \ x_5 = 1,\ \ x_6 = 2,\ \ x_7 = 0.$$